A new proof of the atomic decomposition of Hardy spaces

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Abstract. A new proof is given of the atomic decomposition of Hardy spaces $H^p$, $0 < p \leq 1$, in the classical setting on $\mathbb{R}^n$. The new method can be used to establish atomic decomposition of maximal Hardy spaces in general and nonclassical settings.

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1. Introduction

The study of the real-variable Hardy spaces $H^p$, $0 < p \leq 1$, on $\mathbb{R}^n$ was pioneered by Stein and Weiss [6] and a major step forward in developing this theory was made by Fefferman and Stein in [3], see also [5]. Since then there has been a great deal of work done on Hardy spaces. The atomic decomposition of $H^p$ was first established by Coifman [1] in dimension $n = 1$ and by Latter [4] in dimensions $n > 1$.

The purpose of this article is to give a new proof of the atomic decomposition of the $H^p$ spaces in the classical setting on $\mathbb{R}^n$. Our method does not use the Calderón-Zygmund decomposition of functions and an approximation of the identity as the classical argument does, see [5]. The main advantage of the new proof over the classical one is that it is amenable to utilization in more general and nonclassical settings. For instance, it is used in [2] for establishing the equivalence of maximal and atomic Hardy spaces in the general setting of a metric measure space with the doubling property and in the presence of a non-negative self-adjoint operator whose heat kernel has Gaussian localization.

Notation. We denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^n$ and by $B(x, \delta)$ the open ball centered at $x \in \mathbb{R}^n$ of radius $\delta$, i.e. $B(x, \delta) := \{ y \in \mathbb{R}^n : |x - y| < \delta \}$. Positive constants will be denoted by $c, c_1, \ldots$ and they may vary at every occurrence; $a \sim b$ will stand for $c_1 \leq a/b \leq c_2$. 
1.1. Maximal operators and $H^p$ spaces

We begin by recalling some basic facts about Hardy spaces on $\mathbb{R}^n$. For a complete account of Hardy spaces we refer the reader to [5].

Given $\varphi \in S$ with $S$ being the Schwartz class on $\mathbb{R}^n$ and $f \in S'$ one defines

$$M_\varphi f(x) := \sup_{t > 0} |\varphi_t * f(x)| \quad \text{with} \quad \varphi_t(x) := t^{-n} \varphi(t^{-1} x), \quad \text{and}$$

$$M^*_{\varphi,a} f(x) := \sup_{t > 0} \sup_{y \in \mathbb{R}^n, |x - y| \leq at} |\varphi_t * f(y)|, \quad a \geq 1. \quad (2)$$

We now recall the grand maximal operator. Write

$$P_N(\varphi) := \sup_{x \in \mathbb{R}^n} (1 + |x|)^N \max_{|\alpha| \leq N + 1} |\partial^\alpha \varphi(x)|$$

and denote

$$F_N := \{ \varphi \in S : P_N(\varphi) \leq 1 \}.$$ The grand maximal operator $M_N$ is defined by

$$M_N f(x) := \sup_{\varphi \in F_N} M^*_{\varphi,1} f(x), \quad f \in S'. \quad (3)$$

It is easy to see that for any $\varphi \in S$ and $a \geq 1$ one has

$$M^*_{\varphi,a} f(x) \leq a^N P_N(\varphi) M_N f(x), \quad f \in S'. \quad (4)$$

**Definition 1.** The space $H^p$, $0 < p \leq 1$, is defined as the set of all bounded distributions $f \in S'$ such that the Poisson maximal function $\sup_{t > 0} |P_t * f(x)|$ belongs to $L^p$; the quasi-norm on $H^p$ is defined by

$$\|f\|_{H^p} := \| \sup_{t > 0} |P_t * f(\cdot)| \|_{L^p}. \quad (5)$$

As is well known the following assertion holds, see [3, 5]:

**Proposition 1.** Let $0 < p \leq 1$, $a \geq 1$, and assume $\varphi \in S$ and $\int_{\mathbb{R}^n} \varphi \neq 0$. Then for any $N \geq \lfloor \frac{d}{p} \rfloor + 1$

$$\|f\|_{H^p} \sim \|M^*_{\varphi,a} f\|_{L^p} \sim \|M_N f\|_{L^p}, \quad \forall f \in H^p. \quad (6)$$

1.2. Atomic $H^p$ spaces

A function $a \in L^\infty(\mathbb{R}^n)$ is called an atom if there exists a ball $B$ such that

(i) $\text{supp } a \subset B$,

(ii) $\|a\|_{L^\infty} \leq |B|^{-1/p}$, and
(iii) $\int_{\mathbb{R}^n} x^\alpha a(x)dx = 0$ for all $\alpha$ with $|\alpha| \leq n(p^{-1} - 1)$.

The atomic Hardy space $H^p_A$, $0 < p \leq 1$, is defined as the set of all distributions $f \in S'$ that can be represented in the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \text{where} \quad \sum_{j=1}^{\infty} |\lambda_j|^p < \infty,$$

where $\{a_j\}$ are atoms, and the convergence is in $S'$. Set

$$\|f\|_{H^p_A} := \inf_{f = \sum_j \lambda_j a_j} \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p}, \quad f \in H^p_A.$$

2. Atomic decomposition of $H^p$ spaces

We now come to the main point in this article, that is, to give a new proof of the following classical result [1, 4], see also [5]:

**Theorem 1.** For any $0 < p \leq 1$ the continuous embedding $H^p \subset H^p_A$ is valid, that is, if $f \in H^p$, then $f \in H^p_A$ and

$$\|f\|_{H^p_A} \leq c \|f\|_{H^p},$$

where $c > 0$ is a constant depending only on $p, n$. This along with the easy to prove embedding $H^p_A \subset H^p$ leads to $H^p = H^p_A$ and $\|f\|_{H^p_A} \sim \|f\|_{H^p}$ for $f \in H^p$.

**Proof.** We first derive a simple decomposition identity which will play a central role in this proof. For this construction we need the following

**Lemma 1.** For any $m \geq 1$ there exists a function $\varphi \in C^\infty_0(\mathbb{R}^n)$ such that $\text{supp} \varphi \subset B(0,1)$, $\varphi(0) = 1$, and $\partial^\alpha \varphi(0) = 0$ for $0 < |\alpha| \leq m$. Here $\hat{\varphi}$ is the Fourier transform of $\varphi$, defined by $\hat{\varphi}(\xi) := \int_{\mathbb{R}^n} \varphi(x)e^{-ix\xi}dx$.

**Proof.** We will construct a function $\varphi$ with the claimed properties in dimension $n = 1$. Then a normalized dilation of $\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)$ will have the claimed properties on $\mathbb{R}^n$.

For the univariate construction, pick a smooth “bump” $\phi$ with the following properties: $\phi \in C_0^\infty(\mathbb{R})$, supp $\phi \subset [-1/4, 1/4]$, $\phi(x) > 0$ for $x \in (-1/4, 1/4)$, and $\phi$ is even. Let $\Theta(x) := \phi(x + 1/2) - \phi(x - 1/2)$ for $x \in \mathbb{R}$. Clearly $\Theta$ is odd.

We may assume that $m \geq 1$ is even, otherwise we work with $m + 1$ instead. Denote $\Delta_m^h := (T_h - T_{-h})^m$, where $T_h f(x) := f(x + h)$.

We define $\varphi(x) := \frac{1}{2} \Delta_m^h \Theta(x)$, where $h = \frac{1}{8m}$. Clearly, $\varphi \in C^\infty(\mathbb{R})$, $\varphi$ is even, and supp $\varphi \subset [-\frac{7}{8}, -\frac{1}{8}] \cup [\frac{1}{8}, \frac{7}{8}]$. It is readily seen that for $\nu = 1, 2, \ldots, m$

$$\hat{\varphi}^{(\nu)}(\xi) = (-i)^\nu \int_{\mathbb{R}} x^{\nu-1} \Delta_m^h \Theta(x)e^{-ix\xi}dx$$
Thus we arrive at
\[
\hat{\varphi}^{(\nu)}(0) = (-i)^{\nu} \int_{\mathbb{R}} x^{\nu-1} \Delta_h^{m} \Theta(x) dx = (-i)^{\nu+m} \int_{\mathbb{R}} \Theta(x) \Delta_h^{m} x^{\nu-1} dx = 0.
\]

On the other hand,
\[
\hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x) dx = 2 \int_{0}^{\infty} x^{-1} \Delta_h^{m} \Theta(x) dx = 2(-1)^{m} \int_{1/4}^{3/4} \Theta(x) \Delta_h^{m} x^{-1} dx.
\]

However, if \( f \) a sufficiently smooth function, then \( \Delta_h^{m} f(x) = (2h)^m f^{(m)}(\xi) \), where \( \xi \in (x - mh, x + mh) \). Hence,
\[
\Delta_h^{m} x^{-1} = (2h)^m m!(-1)^m \xi^{-m-1} \text{ with } \xi \in (x - mh, x + mh) \subset [1/8, 7/8].
\]

Consequently, \( \hat{\varphi}(0) \neq 0 \) and then \( \hat{\varphi}^{-1}(0) \varphi(x) \) has the claimed properties. □

With the aid of the above lemma, we pick \( \varphi \in C_0^\infty(\mathbb{R}^n) \) with the following properties: \( \text{supp} \varphi \subset B(0, 1) \), \( \varphi(0) = 1 \), and \( \partial^\alpha \varphi(0) = 0 \) for \( 0 < |\alpha| \leq K \), where \( K \) is sufficiently large. More precisely, we choose \( K \geq n/p \).

Set \( \psi(x) := 2^n \varphi(2x) - \varphi(x) \). Then \( \hat{\psi}(\xi) = \hat{\varphi}(\xi/2) - \hat{\varphi}(\xi) \). Therefore,
\[
\partial^\alpha \hat{\psi}(0) = 0 \text{ for } |\alpha| \leq K \text{ which implies } \int_{\mathbb{R}^n} x^\alpha \psi(x) dx = 0 \text{ for } |\alpha| \leq K.
\]

We also introduce the function \( \hat{\psi}(x) := 2^n \varphi(2x) + \varphi(x) \). We will use the notation \( h_k(x) := 2^{kn}h(2^k x) \).

Clearly, for any \( f \in S' \) we have \( f = \lim_{j \to \infty} \varphi_j \ast f \) (convergence in \( S' \)), which leads to the following representation: For any \( j \in \mathbb{Z} \)
\[
f = \varphi_j \ast f + \sum_{k=j}^{\infty} \left[ \varphi_{k+1} \ast f - \varphi_k \ast f \right] = \varphi_j \ast f + \sum_{k=j}^{\infty} \left[ \varphi_{k+1} - \varphi_k \right] \ast f.
\]

Thus we arrive at
\[
f = \varphi_j \ast f + \sum_{k=j}^{\infty} \psi_k \ast \tilde{\psi}_k \ast f, \quad \forall f \in S', \forall j \in \mathbb{Z} \text{ (convergence in } S'). \tag{10}
\]

Observe that \( \text{supp} \psi_k \subset B(0, 2^{-k}) \) and \( \text{supp} \tilde{\psi}_k \subset B(0, 2^{-k}) \).

In what follows we will utilize the grand maximal operator \( M_N \), defined in (3) with \( N := \left\lfloor \frac{n}{p} \right\rfloor + 1 \). The following claim follows readily from (4): If \( \phi \in S \), then for any \( f \in S' \), \( k \in \mathbb{Z} \), and \( x \in \mathbb{R}^n \)
\[
|\phi_k \ast f(y)| \leq c M_N f(x) \quad \text{for all } y \in \mathbb{R}^n \text{ with } |y - x| \leq 2^{-k+1}, \tag{11}
\]
where the constant \( c > 0 \) depends only on \( P_N(\phi) \) and \( N \).
Let $f \in H^p$, $0 < p \leq 1$, $f \neq 0$. We define

$$\Omega_r := \{x \in \mathbb{R}^n : \mathcal{M}_N f(x) > 2^r \}, \quad r \in \mathbb{Z}. \quad (12)$$

Clearly, $\Omega_r$ is open, $\Omega_{r+1} \subset \Omega_r$, and $\mathbb{R}^n = \bigcup_{r \in \mathbb{Z}} \Omega_r$. It is easy to see that

$$\sum_{r \in \mathbb{Z}} 2^{pr} |\Omega_r| \leq c \int_{\mathbb{R}^n} \mathcal{M}_N f(x)^p dx \leq c \|f\|_{L^p}^p. \quad (13)$$

Indeed, we have

$$\sum_{r \in \mathbb{Z}} 2^{pr} |\Omega_r| = \sum_{r \in \mathbb{Z}} 2^{pr} \sum_{\nu \geq r} |\Omega_{\nu} \setminus \Omega_{\nu+1}| = \sum_{\nu \in \mathbb{Z}} \sum_{r \leq \nu} 2^{pr} |\Omega_{\nu} \setminus \Omega_{\nu+1}| \leq c_p \sum_{\nu \in \mathbb{Z}} \int_{\Omega_{\nu} \setminus \Omega_{\nu+1}} \mathcal{M}_N f(x)^p dx$$

$$= c_p \int_{\mathbb{R}^n} \mathcal{M}_N f(x)^p dx.$$

From (13) we get $|\Omega_r| \leq c 2^{-pr} \|f\|_{L^p}^p$ for $r \in \mathbb{Z}$. Therefore, for any $r \in \mathbb{Z}$ there exists $J > 0$ such that $\|\varphi_j * \varphi_j * f\|_{L^\infty} \leq c 2^r$ for $j < -J$. Consequently, $\|\varphi_j * \varphi_j * f\|_{L^\infty} \to 0$ as $j \to -\infty$, which implies

$$f = \lim_{K \to \infty} \sum_{k = -\infty}^{K} \psi_k \ast \tilde{\psi}_k \ast f \quad \text{(convergence in } S'). \quad (14)$$

Assuming that $\Omega_r \neq \emptyset$ we write

$$E_{rk} := \{x \in \Omega_r : \text{dist}(x, \Omega_{r+1}) > 2^{-k+1} \} \setminus \{x \in \Omega_{r+1} : \text{dist}(x, \Omega_{r+1}) > 2^{-k+1} \}.$$  

By (13) it follows that $|\Omega_r| < \infty$ and hence there exists $s_r \in \mathbb{Z}$ such that $E_{rs_r} \neq \emptyset$ and $E_{rk} = \emptyset$ for $k < s_r$. Evidently $s_r \leq s_{r+1}$. We define

$$F_r(x) := \sum_{k \geq s_r} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy, \quad x \in \mathbb{R}^n, \quad r \in \mathbb{Z}, \quad (15)$$

and more generally

$$F_{r, \kappa_0, \kappa_1}(x) := \sum_{k = \kappa_0}^{\kappa_1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy, \quad s_r \leq \kappa_0 \leq \kappa_1 \leq \infty. \quad (16)$$

It will be shown in Lemma 2 below that the functions $F_r$ and $F_{r, \kappa_0, \kappa_1}$ are well defined and $F_r, F_{r, \kappa_0, \kappa_1} \in L^\infty$.

Note that $\text{supp } \psi_k \subset B(0,2^{-k})$ and hence

$$\text{supp } \left( \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy \right) \subset \{x : \text{dist}(x,E_{rk}) < 2^{-k} \}. \quad (17)$$
On the other hand, clearly $B(y, 2^{−k+1}) \cap (\Omega_r \setminus \Omega_{r+1}) \neq \emptyset$ for each $y \in E_{rk}$, and $\mathcal{P}_N(\tilde{\psi}) \leq c$. Therefore, see (11), $|\tilde{\psi}_k * f(y)| \leq c 2^r$ for $y \in E_{rk}$, which implies

$$\left\| \int_\mathbb{R} \psi_k(\cdot - y)\tilde{\psi}_k * f(y) dy \right\|_\infty \leq c 2^r, \quad \forall E \subset E_{rk}. \quad (18)$$

Similarly,

$$\left\| \int_\mathbb{R} \varphi_k(\cdot - y)\tilde{\varphi}_k * f(y) dy \right\|_\infty \leq c 2^r, \quad \forall E \subset E_{rk}. \quad (19)$$

We collect all we need about the functions $F_r$ and $F_{r, \kappa_0, \kappa_1}$ in the following

**Lemma 2.** (a) We have

$$E_{rk} \cap E_{r'k} = \emptyset \text{ if } r \neq r' \text{ and } \mathbb{R}^n = \bigcup_{r \in \mathbb{Z}} E_{rk}, \quad \forall k \in \mathbb{Z}. \quad (20)$$

(b) There exists a constant $c > 0$ such that for any $r \in \mathbb{Z}$ and $s_r \leq \kappa_0 \leq \kappa_1 \leq \infty$

$$\|F_r\|_\infty \leq c 2^r, \quad \|F_{r, \kappa_0, \kappa_1}\|_\infty \leq c 2^r. \quad (21)$$

(c) The series in (15) and (16) (if $\kappa_1 = \infty$) converge point-wise and in distributional sense.

(d) Moreover,

$$F_r(x) = 0, \quad \forall x \in \mathbb{R}^n \setminus \Omega_r, \quad \forall r \in \mathbb{Z}. \quad (22)$$

**Proof.** Identities (20) are obvious and (22) follows readily from (17).

We next prove the left-hand side inequality in (21); the proof of the right-hand side inequality is similar and will be omitted. Consider the case when $\Omega_{r+1} \neq \emptyset$ (the case when $\Omega_{r+1} = \emptyset$ is easier). Write

$$U_k := \{x \in \Omega_r : \text{dist}(x, \Omega_r^c) > 2^{-k+1}\},$$
$$V_k := \{x \in \Omega_{r+1} : \text{dist}(x, \Omega_{r+1}^c) > 2^{-k+1}\}.$$ 

Observe that $E_{rk} = U_k \setminus V_k$.

Clearly, (17) implies $|F_r(x)| = 0$ for $x \in \mathbb{R}^n \setminus \bigcup_{k \geq s_r} \{x : \text{dist}(x, E_{rk}) < 2^{-k}\}$.

We next estimate $|F_r(x)|$ for $x \in \bigcup_{k \geq s_r} \{x : \text{dist}(x, E_{rk}) < 2^{-k}\}$. Two cases present themselves here.

**Case 1:** $x \in \bigcap_{k \geq s_r} \{x : \text{dist}(x, E_{rk}) < 2^{-k}\} \cap \Omega_{r+1}$. Then there exist $\nu, \ell \in \mathbb{Z}$ such that

$$x \in (U_{\ell+1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_\nu). \quad (23)$$

Due to $\Omega_{r+1} \subset \Omega_r$ we have $V_k \subset U_k$, implying $(U_{\ell+1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_\nu) = \emptyset$ if $\nu < \ell$. We next consider two subcases depending on whether $\nu \geq \ell + 3$ or $\ell \leq \nu \leq \ell + 2$.

(a) Let $\nu \geq \ell + 3$. We claim that (23) yields

$$B(x, 2^{-k}) \cap E_{rk} = \emptyset \text{ for } k \geq \nu + 2 \text{ or } k \leq \ell - 1. \quad (24)$$
Indeed, if \( k \geq \nu + 2 \), then \( E_{rk} \subset \Omega_r \setminus V_{\nu + 2} \), which implies (24), while if \( k \leq \ell - 1 \), then \( E_{rk} \subset U_{\ell - 1} \), again implying (24).

We also claim that

\[
B(x, 2^{-k}) \subset E_{rk} \quad \text{for} \quad \ell + 2 \leq k \leq \nu - 1.
\]

Indeed, clearly

\[
(U_{\ell + 1} \setminus U_{\ell}) \cap (V_{\nu + 1} \setminus V_{\nu}) \subset (U_{k - 1} \setminus U_{\ell}) \cap (V_{\nu + 1} \setminus V_{\nu + 1}) \subset U_{k - 1} \setminus V_{k + 1},
\]

which implies (25).

From (17) and (24)- (25) it follows that

\[
F_r(x) = \sum_{k=\ell}^{\nu + 1} \int_{E_{rk}} \psi_k(x - y) \tilde{\psi}_k * f(y) dy = \sum_{k=\ell}^{\nu + 1} \int_{E_{rk}} \psi_k(x - y) \psi_k * f(y) dy
\]

\[
+ \sum_{k=\ell + 2}^{\nu - 2} \int_{\mathbb{R}^n} \psi_k(x - y) \tilde{\psi}_k * f(y) dy + \sum_{k=\nu - 1}^{\nu + 1} \int_{E_{rk}} \psi_k(x - y) \tilde{\psi}_k * f(y) dy.
\]

However,

\[
\sum_{k=\ell + 2}^{\nu - 2} \int_{\mathbb{R}^n} \psi_k(x - y) \tilde{\psi}_k * f(y) dy
\]

\[
= \sum_{k=\ell + 2}^{\nu - 2} [\varphi_{k + 1} * \varphi_{k + 1} * f(x) - \varphi_k * \varphi_k * f(x)]
\]

\[
= \varphi_{\nu - 1} * \varphi_{\nu - 1} * f(x) - \varphi_{\ell + 2} * \varphi_{\ell + 2} * f(x)
\]

\[
= \int_{E_{r, \nu - 1}} \varphi_{\nu - 1}(x - y) \varphi_{\nu - 1} * f(y) dy - \int_{E_{r, \ell + 2}} \varphi_{\ell + 2}(x - y) \varphi_{\ell + 2} * f(y) dy.
\]

Combining the above with (18) and (19) we obtain \( |F_r(x)| \leq c 2^r \).

(b) Let \( \ell \leq \nu \leq \ell + 2 \). Just as above we have

\[
F_r(x) = \sum_{k=\ell}^{\nu + 1} \int_{E_{rk}} \psi_k(x - y) \tilde{\psi}_k * f(y) dy = \sum_{k=\ell}^{\nu + 1} \int_{E_{rk}} \psi_k(x - y) \tilde{\psi}_k * f(y) dy.
\]

We use (18) to estimate each the above four integrals and obtain \( |F_r(x)| \leq c 2^r \).

**Case 2:** \( x \in \Omega_r \setminus \Omega_{r + 1} \). Then there exists \( \ell \geq s \), such that

\[
x \in (U_{\ell + 1} \setminus U_{\ell}) \cap (\Omega_r \setminus \Omega_{r + 1}).
\]

Just as in the proof of (24) we have \( B(x, 2^{-k}) \subset E_{rk} = \emptyset \) for \( k \leq \ell - 1 \), and as in the proof of (25) we have

\[
(U_{\ell + 1} \setminus U_{\ell}) \cap (\Omega_r \setminus \Omega_{r + 1}) \subset U_{k - 1} \setminus V_{k + 1}.
\]
which implies $B(x, 2^{-k}) \subset E_{r_k}$ for $k \geq \ell + 2$. We use these and (17) to obtain

$$F_r(x) = \sum_{k=\ell}^{\infty} \int_{E_{r_k}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy$$

$$= \sum_{k=\ell}^{\ell+1} \int_{E_{r_k}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy + \sum_{k=\ell+2}^{\infty} \int_{\mathbb{R}^n} \psi_k(x-y)\tilde{\psi}_k * f(y)dy.$$

For the last sum we have

$$\sum_{k=\ell+2}^{\infty} \int_{\mathbb{R}^n} \psi_k(x-y)\tilde{\psi}_k * f(y)dy = \lim_{\nu \to \infty} \sum_{k=\ell+2}^{\nu} \psi_k * \tilde{\psi}_k * f(x)$$

$$= \lim_{\nu \to \infty} \left( \int_{E_{r,\nu+1}} \varphi_{\nu+1}(x-y)\tilde{\varphi}_{\nu+1} * f(y)dy \right)$$

$$- \int_{E_{r,\ell+2}} \varphi_{\ell+2}(x-y)\tilde{\varphi}_{\ell+2} * f(y)dy.$$

From the above and (18)-(19) we obtain $|F_r(x)| \leq c 2^r$.

The point-wise convergence of the series in (15) follows from above and we similarly establish the point-wise convergence in (16).

The convergence in distributional sense in (15) relies on the following assertion: For every $\phi \in \mathcal{S}$

$$\sum_{k \geq s} |\langle g_{r_k}, \phi \rangle| < \infty, \text{ where } g_{r_k}(x) := \int_{E_{r_k}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy. \quad (26)$$

Here $\langle g_{r_k}, \phi \rangle := \int_{\mathbb{R}^n} g_{r_k} \overline{\phi} dx$. To prove the above we will employ this estimate:

$$\|\tilde{\psi}_k f\|_{\infty} \leq c 2^{kn/p} \|f\|_{H^p}, \quad k \in \mathbb{Z}. \quad (27)$$

Indeed, using (4) we get

$$|\tilde{\psi}_k f(x)|^p \leq \inf_{y: |x-y| \leq 2^{-k}} \sup_{z: |y-z| \leq 2^{-k}} |\tilde{\psi}_k f(z)|^p \leq \inf_{y: |x-y| \leq 2^{-k}} c M_N(f)(y)^p$$

$$\leq c |B(x, 2^{-k})|^{-1} \int_{B(x, 2^{-k})} M_N(f)(y)^p dy \leq c 2^{kn} \|f\|_{H^p}^p,$$

and (27) follows.

We will also need the following estimate: For any $\sigma > n$ there exists a constant $c_\sigma > 0$ such that

$$\int_{\mathbb{R}^n} |\psi_k(x-y)\phi(x)| dx \leq c_\sigma 2^{-k(K+1)} (1 + |y|)^{\sigma}, \quad y \in \mathbb{R}^n, k \geq 0. \quad (28)$$
This is a standard estimate for inner products taking into account that \( \phi \in \mathcal{S} \) and \( \psi \in C^\infty \), \( \text{supp} \, \psi \subset B(0,1) \), and \( \int_{\mathbb{R}^n} x^\alpha \psi(x)dx = 0 \) for \( |\alpha| \leq K \).

We now estimate \( \langle g_{r,k}, \phi \rangle \). From (27) and the fact that \( \psi \in C_0^\infty(\mathbb{R}) \) and \( \phi \in \mathcal{S} \) it readily follows that

\[
\int_{E_{r,k}} \int_{\mathbb{R}^n} |\psi_k(x-y)||\phi(x)||\tilde{\psi}_k f(y)|dydx < \infty, \quad k \geq s_r.
\]

Therefore, we can use Fubini’s theorem, (27), and (28) to obtain for \( k \geq 0 \)

\[
|\langle g_{r,k}, \phi \rangle| \leq \int_{E_{r,k}} \int_{\mathbb{R}^n} \psi_k(x-y)\phi(x)dx \left| \tilde{\psi}_k f(y) \right|dy \\
\leq e^{-k(K+1-n/p)}\|f\|_{H^p} \int_{E_{r,k}} (1 + |y|)^{-\sigma}dy \leq e^{-k(K+1-n/p)}\|f\|_{H^p},
\]

which implies (26) because \( K \geq n/p \).

Denote \( G_{\ell} := \sum_{k=s_{\ell}}^{\ell} g_{rk} \). From the above proof of (b) and (21) we infer that \( G_\ell(x) \to F_r(x) \) as \( \ell \to \infty \) for \( x \in \mathbb{R}^n \) and \( \|G_\ell\| \leq e^{2^\ell} < \infty \) for \( \ell \geq s_r \).

On the other hand, from (26) it follows that the series \( \sum_{k \geq s_r} g_{rk} \) converges in distributional sense. By applying the dominated convergence theorem one easily concludes that \( F_r = \sum_{k \geq s_r} g_{rk} \) with the convergence in distributional sense.

We set \( F_r := 0 \) in the case when \( \Omega_r = \emptyset, r \in \mathbb{Z} \).

Note that by (20) it follows that

\[
\psi_k \ast \tilde{\psi}_k \ast f(x) = \int_{\mathbb{R}^n} \psi_k(x-y)\tilde{\psi}_k \ast f(y)dy = \sum_{r \in \mathbb{Z}} \int_{E_{r,k}} \psi_k(x-y)\tilde{\psi}_k \ast f(y)dy
\]

and using (14) and the definition of \( F_r \) in (15) we arrive at

\[
f = \sum_{r \in \mathbb{Z}} F_r \quad \text{in} \quad \mathcal{S}', \quad \text{i.e.} \quad \langle f, \phi \rangle = \sum_{r \in \mathbb{Z}} \langle F_r, \phi \rangle, \quad \forall \phi \in \mathcal{S},
\]

where the last series converges absolutely. Above \( \langle f, \phi \rangle \) denotes the action of \( f \) on \( \mathcal{S}' \). We next provide the needed justification of equality (31).

From (14), (15), (30), and the notation from (26) we obtain for \( \phi \in \mathcal{S} \)

\[
\langle f, \phi \rangle = \sum_k \langle \psi_k \ast \tilde{\psi}_k \ast f, \phi \rangle = \sum_k \sum_r \langle g_{rk}, \phi \rangle = \sum_r \sum_k \langle g_{rk}, \phi \rangle = \sum_r \langle F_r, \phi \rangle.
\]

Clearly, to justify the above it suffices to show that \( \sum_k \sum_r |\langle g_{rk}, \phi \rangle| < \infty \). We split this sum into two:

\[
\sum_k \sum_r |\langle g_{rk}, \phi \rangle| = \sum_{k \geq 0} \sum_r |\langle g_{rk}, \phi \rangle| + \sum_{k < 0} \sum_r |\langle g_{rk}, \phi \rangle| =: \Sigma_1 + \Sigma_2.
\]
To estimate $\Sigma_1$ we use (29) and obtain

$$
\Sigma_1 \leq c\|f\|_{H^p} \sum_{k \geq 0} 2^{-k(K+1-n/p)} \int_{E_k} (1 + |y|)^{-\sigma} dy
$$

Here we also used that $K \geq n/p$ and $\sigma > n$.

We estimate $\Sigma_2$ in a similar manner, using that $\int_{\mathbb{R}^n} |\psi_k(y)| dy \leq c < \infty$ and (27). We get

$$
\Sigma_2 \leq c\|f\|_{H^p} \sum_{k \leq 0} 2^{kn/p} \int_{\mathbb{R}^n} (1 + |x|)^{n-1} dx \leq c\|f\|_{H^p}.
$$

The above estimates of $\Sigma_1$ and $\Sigma_2$ imply $\sum_k \left| \langle g_{rk}, \phi \rangle \right| < \infty$, which completes the justification of (31).

Observe that due to $\int_{\mathbb{R}^n} x^\alpha \psi(x) dx = 0$ for $|\alpha| \leq K$ we have

$$
\int_{\mathbb{R}^n} x^\alpha F_r(x) dx = 0 \quad \text{for } |\alpha| \leq K, \ r \in \mathbb{Z}. \quad (32)
$$

We next decompose each function $F_r$ into atoms. To this end we need a Whitney type cover for $\Omega_r$, given in the following

**Lemma 3.** Suppose $\Omega$ is an open proper subset of $\mathbb{R}^n$ and let $\rho(x) := \text{dist}(x, \Omega^c)$. Then there exists a constant $K > 0$, depending only on $n$, and a sequence of points $\{\xi_j\}_{j \in \mathbb{N}}$ in $\Omega$ with the following properties, where $\rho_j := \text{dist}(\xi_j, \Omega^c)$:

1. $\Omega = \bigcup_{j \in \mathbb{N}} B(\xi_j, \rho_j/2)$.
2. $\{B(\xi_j, \rho_j/5)\}$ are disjoint.
3. If $B(\xi_j, 3\rho_j/4) \cap B(\xi_\nu, 3\rho_\nu/4) \neq \emptyset$, then $7^{-1}\rho_\nu \leq \rho_j \leq 7\rho_\nu$.
4. For every $j \in \mathbb{N}$ there are at most $K$ balls $B(\xi_j, 3\rho_j/4)$ intersecting $B(\xi_j, 3\rho_j/4)$.

Variants of this simple lemma are well known and frequently used. To prove it one simply selects $\{B(\xi_j, \rho(\xi_j)/5)\}_{j \in \mathbb{N}}$ to be a maximal disjoint subset of $\{B(x, \rho(x)/5)\}_{x \in \Omega}$ and then properties (a)-(d) follow readily, see [5], pp. 15-16.

We apply Lemma 3 to each set $\Omega_r \neq \emptyset, r \in \mathbb{Z}$. Fix $r \in \mathbb{Z}$ and assume $\Omega_r \neq \emptyset$. Denote by $B_j := B(\xi_j, \rho_j/2), j = 1, 2, \ldots$, the balls given by Lemma 3, applied
to $\Omega_r$, with the additional assumption that these balls are ordered so that $\rho_1 \geq \rho_2 \geq \cdots$. We will adhere to the notation from Lemma 3. We will also use the more compact notation $B_r := \{B_j\}_{j \in \mathbb{N}}$ for the set of balls covering $\Omega_r$.

For each ball $B \in B_r$ and $k \geq s_r$ we define
\[
E_{\ell}^{B, r}: = \begin{cases} E_{r, k} \cap \{x : \text{dist}(x, B) < 2^{-k+1}\} & \text{if } B \cap E_{r, k} \neq \emptyset \quad (33) \\
\emptyset & \text{if } B \cap E_{r, k} = \emptyset.
\end{cases}
\]

and set $E_{\ell}^{B, r} := \emptyset$ if $B \cap E_{r, k} = \emptyset$.

We also define, for $\ell = 1, 2, \ldots$,
\[
R_{\ell}^{B, r} := E_{\ell}^{B, r} \setminus \cup_{\nu > \ell} E_{\ell}^{B, r} \quad (34)
\]
\[
F_{\ell}^{B, r}(x) := \sum_{k \geq s_r} \int_{R_{\ell}^{B, r}} \psi_k(x - y) \tilde{\psi}_k * f(y) dy.
\quad (35)
\]

**Lemma 4.** For every $\ell \geq 1$ the function $F_{\ell}^{B, r}$ is well defined, more precisely, the series in (35) converges point-wise and in distributional sense. Furthermore,
\[
\text{supp } F_{\ell}^{B, r} \subset 7B_{\ell}, \quad (36)
\]
\[
\int_{\mathbb{R}^n} x^{\alpha} F_{\ell}^{B, r}(x) dx = 0 \quad \text{for all } \alpha \text{ with } |\alpha| \leq n(p^{-1} - 1), \quad (37)
\]
and
\[
\|F_{\ell}^{B, r}\|_\infty \leq c_2 2^r, \quad (38)
\]
where the constant $c_2$ is independent of $r, \ell$.

In addition, for any $k \geq s_r$
\[
E_{r, k} = \cup_{\ell \geq 1} R_{\ell}^{B, r} \quad \text{and} \quad R_{r, k}^{B, r} \cap R_{r, k}^{B, m} = \emptyset, \quad \ell \neq m. \quad (39)
\]
Hence
\[
F_r = \sum_{B \in B_r} F_B \quad (\text{convergence in } S'). \quad (40)
\]

**Proof.** Fix $\ell \geq 1$. Observe that using Lemma 3 we have
\[
B_\ell \subset \{x \in \mathbb{R}^n : \text{dist}(x, \Omega_\ell^c) < 2\rho_\ell\}
\]
and hence $E_{r, k}^{B, r} := \emptyset$ if $2^{-k+1} \geq 2\rho_\ell$. Define $k_0 := \min\{k : 2^{-k} < \rho_\ell\}$. Hence $\rho_\ell/2 \leq 2^{-k_0} < \rho_\ell$. Consequently,
\[
F_{\ell}^{B, r}(x) := \sum_{k \geq k_0} \int_{R_{\ell}^{B, r}} \psi_k(x - y) \tilde{\psi}_k * f(y) dy. \quad (41)
\]

It follows that $\text{supp } F_{\ell}^{B, r} \subset B(\xi_\ell, (7/2)\rho_\ell) = 7B_{\ell}$, which confirms (36).

To prove (38) we will use the following
Lemma 5. For an arbitrary set \( S \subset \mathbb{R}^n \) let
\[
S_k := \{ x \in \mathbb{R}^n : \text{dist}(x, S) < 2^{-k+1} \}
\]
and set
\[
F_S(x) := \sum_{k \geq \kappa_0} \int_{E_{rk} \cap S_k} \psi_k(x - y) \tilde{\psi}_k * f(y) \, dy
\]
for some \( \kappa_0 \geq s_r \). Then \( \| F_S \|_\infty \leq c 2^r \), where \( c > 0 \) is a constant independent of \( S \) and \( \kappa_0 \). Moreover, the above series converges in \( S' \).

Proof. From (17) it follows that \( F_S(x) = 0 \) if \( \text{dist}(x, S) \geq 3 \times 2^{-\kappa_0} \).

Let \( x \in S \). Evidently, \( B(x, 2^{-k}) \subset S_k \) for every \( k \) and hence
\[
F_S(x) = \sum_{k \geq \kappa_0} \int_{E_{rk} \cap B(x, 2^{-k})} \psi_k(x - y) \tilde{\psi}_k * f(y) \, dy
\]
and hence
\[
\sum_{k \geq \kappa_0} \int_{E_{rk} \cap S_k} \psi_k(x - y) \tilde{\psi}_k * f(y) \, dy = F_{r, \kappa_0}(x).
\]

On account of Lemma 2 (b) we obtain \( |F_S(x)| = |F_{r, \kappa_0}(x)| \leq c 2^r \).

Consider the case when \( x \in S_l \setminus S_{l+1} \) for some \( \ell \geq \kappa_0 \). Then \( B(x, 2^{-k}) \subset S_k \) if \( \kappa_0 \leq k \leq \ell - 1 \) and \( B(x, 2^{-k}) \cap S_k = \emptyset \) if \( k \geq \ell + 2 \). Therefore,
\[
F_S(x) = \sum_{k=\kappa_0}^{\ell-1} \int_{E_{rk}} \psi_k(x - y) \tilde{\psi}_k * f(y) \, dy + \sum_{k=\ell}^{\ell+1} \int_{E_{rk} \cap S_k} \psi_k(x - y) \tilde{\psi}_k * f(y) \, dy
\]
where we used the notation from (16). By Lemma 2 (b) and (18) it follows that \( |F_S(x)| \leq c 2^r \).

We finally consider the case when \( 2^{-\kappa_0+1} \leq \text{dist}(x, S) < 3 \times 2^{-\kappa_0} \). Then we have \( F_S(x) = \int_{E_{rk} \cap S_0} \psi_{\kappa_0}(x - y) \tilde{\psi}_{\kappa_0} * f(y) \, dy \) and the estimate \( |F_S(x)| \leq c 2^r \) is immediate from (18).

The convergence in \( S' \) in (42) is established as in the proof of Lemma 2. \( \square \)

Fix \( \ell \geq 1 \) and let \( \{ B_j : j \in J \} \) be the set of all balls \( B_j = B(\xi_j, \rho_j/2) \) such that \( j > \ell \) and
\[
B\left(\xi_j, \frac{3\rho_j}{4}\right) \cap B\left(\xi_\ell, \frac{3\rho_\ell}{4}\right) \neq \emptyset.
\]
By Lemma 3 it follows that \( \#J \leq K \) and \( 7^{-1}\rho_\ell \leq \rho_j \leq 7\rho_\ell \) for \( j \in J \). Define
\[
k_1 := \min \left\{ k : 2^{-k+1} < 4^{-1} \min \left\{ \rho_j : j \in J \cup \{ \ell \} \right\} \right\}.
\]
From this definition and \( 2^{-k_0} < \rho_\ell \) we infer
\[
2^{-k_1+1} \geq 8^{-1} \min \left\{ \rho_j : j \in J \cup \{ \ell \} \right\} > 8^{-2} \rho_\ell > 8^{-2} 2^{-k_0} \implies k_1 \leq k_0 + 7.
\]
Clearly, from (43)
\[ \{ x : \text{dist}(x, B_j) < 2^{-k+1} \} \subset B(\xi_j, 3\rho_j/4), \quad \forall k \geq k_1, \quad \forall j \in \mathcal{J} \cup \{ \ell \}. \]  
(45)

Denote \( S := \cup_{j \in \mathcal{J}} B_j \) and \( \tilde{S} := \cup_{j \in \mathcal{J}} B_j \cup B_{\ell} = S \cup B_{\ell}. \) As in Lemma 5 we set \( S_k := \{ x \in \mathbb{R}^n : \text{dist}(x, S) < 2^{-k+1} \} \) and \( \tilde{S}_k := \{ x \in \mathbb{R}^n : \text{dist}(x, \tilde{S}) < 2^{-k+1} \}. \)

It readily follows from the definition of \( k_1 \) in (43) that
\[ R_{r_k}^{B_{r_k}} := E_{r_k}^{B_{r_k}} \setminus \cup_{\nu > \ell} E_{r_k}^{B_{r_k}} = (E_{r_k} \cap \tilde{S}_k) \setminus (E_{r_k} \cap S_k) \quad \text{for} \quad k \geq k_1. \]  
(46)

Denote
\[ F_S(x) := \sum_{k \geq k_1} \int_{E_{r_k} \cap S_k} \psi_k(x-y)\tilde{\psi}_k * f(y) dy, \quad \text{and} \]
\[ F_{\tilde{S}}(x) := \sum_{k \geq k_1} \int_{E_{r_k} \cap \tilde{S}_k} \psi_k(x-y)\tilde{\psi}_k * f(y) dy. \]

From (46) and the fact that \( S \subset \tilde{S} \) it follows that
\[ F_{B_{r_k}}(x) = F_{\tilde{S}}(x) - F_S(x) + \sum_{k_0 \leq k < k_1} \int_{R_{r_k}^{B_{r_k}}} \psi_k(x-y)\tilde{\psi}_k * f(y) dy. \]

By Lemma 5 we get \( \|F_S\|_\infty \leq c2^r \) and \( \|F_{\tilde{S}}\|_\infty \leq c2^r. \) On the other hand from (44) we have \( k_1 - k_0 \leq 7. \) We estimate each of the (at most 7) integrals above using (18) to conclude that \( \|F_{B_{r_k}}\|_\infty \leq c2^r. \)

We deal with the convergence in (35) and (40) as in the proof of Lemma 2.

Clearly, (37) follows from the fact that \( \int_{\mathbb{R}^n} x^\alpha \psi(x) dx = 0 \) for all \( \alpha \) with \( |\alpha| \leq K. \)

Finally, from Lemma 3 we have \( \Omega_r \subset \cup_{j \in \mathbb{N}} B_j \) and then (39) is immediate from (33) and (34). \( \square \)

We are now prepared to complete the proof of Theorem 1. For every ball \( B \in B_r, \) \( r \in \mathbb{Z}, \) provided \( \Omega_r \neq \emptyset, \) we define \( B^* := 7B, \)
\[ a_B(x) := c_t^{-1}|B^*|^{-1/p}2^{-r}F_B(x) \quad \text{and} \quad \lambda_B := c_t|B^*|^{1/p}2^r, \]
where \( c_t > 0 \) is the constant from (38). By (36) \( \text{supp} a_B \subset B^* \) and by (38)
\[ \|a_B\|_\infty \leq c_t^{-1}|B^*|^{-1/p}2^{-r}\|F_B\|_\infty \leq |B^*|^{-1/p}. \]

Furthermore, from (37) it follows that \( \int_{\mathbb{R}^n} x^\alpha a_B(x) dx = 0 \) if \( |\alpha| \leq n(p^{-1} - 1). \)
Therefore, each \( a_B \) is an atom for \( H^p. \)

We set \( B_r := \emptyset \) if \( \Omega_r = \emptyset. \) Now, using the above, (31), and Lemma 4 we get
\[ f = \sum_{r \in \mathbb{Z}} F_r = \sum_{r \in \mathbb{Z}} \sum_{B \in B_r} F_B = \sum_{r \in \mathbb{Z}} \sum_{B \in B_r} \lambda_B a_B, \]
where the convergence is in $S'$, and
\[
\sum_{r \in \mathbb{Z}} \sum_{B \in \mathcal{B}_r} |\lambda_B|^p \leq c \sum_{r \in \mathbb{Z}} 2^{rp} \sum_{B \in \mathcal{B}_r} |B| = c \sum_{r \in \mathbb{Z}} 2^{rp} |\Omega_r| \leq c \|f\|_{H^p}^p,
\]
which is the claimed atomic decomposition of $f \in H^p$. Above we used that $|B^*| = |7B| = 7^n|B|$. The proof of Theorem 1 is complete. \(\square\)

Remark. The proof of Theorem 1 can be considerably simplified and shortened if one seeks to establish atomic decomposition of the $H^p$ spaces in terms of $q$-atoms with $p < q < \infty$ rather than $\infty$-atoms as in Theorem 1, i.e. atoms satisfying $\|a\|_{L^q} \leq |B|^{1/q - 1/p}$ with $q < \infty$ rather than $\|a\|_{L^\infty} \leq |B|^{-1/p}$. We will not elaborate on this here.

References


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