Introduction to function spaces - lecture notes:
Semester B (Spring 2021)
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Banach Spaces

Definition Banach space is a complete normed vector space B over a field $F=\{\mathbb{R},\mathbb{C}\}$,

Vector space: $\exists 0\in B$, $\forall f,g\in B$, $\alpha,\beta\in F\Rightarrow \alpha f+\beta g\in B$.
Complete: Every Cauchy sequence in B converges to an element of B.
Norm:

i. $f\neq 0\Rightarrow \|f\|>0$
ii. $\|\alpha f\|=|\alpha|\|f\|,\qquad \forall\alpha\in F$,
iii. Triangle inequality $\|f+g\|\le\|f\|+\|g\|$

Measure

In this course we shall mostly use the standard Lebesgue measure – the volume of a (measurable) set.

Examples: $\Omega=[0,2]^n\subset\mathbb{R}^n$, $\mu(\Omega)=|\Omega|=2^n$.

But in some cases we will use the notation of 'abstract' measure space. That is (X,μ), where for a measurable $E\subseteq X$, $\mu(E)$ is the measure ('volume'), and for measurable function $f:X\to\mathbb{C}$, we can evaluate $\int_E f(x)\,d\mu(x)$.

Example for weight measure: Let $w:\mathbb{R}^n\to\mathbb{R}_+$, $\int_{\mathbb{R}^n}w(x)\,dx=1$. Then we can define $d\mu(x):=w(x)\,dx$.

We will need the notion of zero measure (volume). Example: a set of discrete points.

In the course we will also study distributions. These are linear functionals on 'smooth' functions. For example, the **Dirac**:

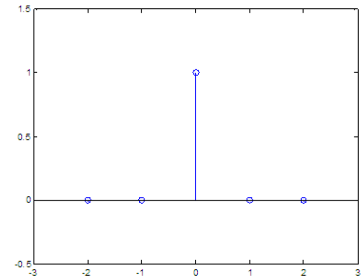
$$\langle\delta_{x_0},f\rangle=\delta_{x_0}(f):=f(x_0),\qquad \forall x_0\in\mathbb{R}^n,\ f\in C(\mathbb{R}^n).$$

Sometimes the Dirac is regarded as a 'function' with value ∞ at x_0. This is misleading and not well defined. It is indeed the 'limit' of a sequence $g_t(x):=a_t e^{-\frac{|x|^2}{t}}$, $\int g_t=1$, $t\to 0$, but as functionals $\langle g_t,f\rangle\underset{t\to 0}{\to}\langle\delta_0,f\rangle$.

Radon measure – compatible with topology of space

i. σ-measurable on Borel sets,
ii. locally finite (every point has a neighborhood of finite measure),
iii. inner regular (measure of a set can be approximated by measure of compact sets)

Lp Spaces



$\Omega \subseteq \mathbb{R}^n$ domain. Examples: $\Omega = [a,b] \subset \mathbb{R}$, $\Omega = [0,1]^n \subset \mathbb{R}^n, \Omega = \mathbb{R}^n$.

$$\|f\|_{L_p(\Omega)} := \begin{cases} \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}, & 0 < p < \infty, \\ \operatorname{ess\,sup}_{x \in \Omega} |f(x)|, & p = \infty. \end{cases}$$

$$\operatorname{ess\,sup}_x |f(x)| := \sup_{A>0} \left\{ A > 0 : \ \left| \{ x : |f(x)| \geq A \} \right| > 0 \right\}.$$

$1 \leq p \leq \infty$ Banach spaces

$0 < p < 1$ Quasi-Banach spaces (quasi-triangle inequality holds)

$$\|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p .$$

Theorem [Hölder] $1 \leq p \leq \infty$, $f \in L_p(\Omega), g \in L_{p'}(\Omega)$

$$\left| \int_\Omega fg \right| \leq \int_\Omega |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_{p'} \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Lemma Young's inequality for products,

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \ \forall a,b \geq 0 .$$

Proof of lemma The logarithmic function is concave. Therefore

$$\begin{aligned} \log\left(\frac{1}{p} a^p + \frac{1}{p'} b^{p'} \right) &= \log\left(\frac{1}{p} a^p + \left(1 - \frac{1}{p}\right) b^{p'} \right) \\ &\geq \frac{1}{p} \log\left(a^p\right) + \frac{1}{p'} \log\left(b^{p'}\right) \\ &= \log(a) + \log(b) = \log(ab). \end{aligned}$$

Since the logarithmic function is increasing, we are done (or we take exp on both sides). □

Proof of theorem If $p = \infty$

$$\int_\Omega |fg| \leq \|f\|_\infty \int_\Omega |g| \leq \|f\|_\infty \|g\|_1 .$$

The proof is similar for $p = 1$. So, assume now $1 < p < \infty$, $\|f\|_p = \|g\|_{p'} = 1$.

Integrating pointwise and applying Young's inequality almost everywhere, gives

$$\int_\Omega |f(x)g(x)|dx \le \int_\Omega \left(\frac{|f(x)|^p}{p} + \frac{|g(x)|^{p'}}{p'} \right) dx$$
$$= \frac{1}{p}\int_\Omega |f(x)|^p\, dx + \frac{1}{p'}\int_\Omega |g(x)|^{p'}\, dx$$
$$= \frac{1}{p} + \frac{1}{p'} = 1.$$

Now assuming $f, g \neq 0$ (else, we're done)

$$\int_\Omega \frac{|f(x)|}{\|f\|_p}\frac{|g(x)|}{\|g\|_{p'}}dx \le 1 \Rightarrow \int_\Omega |fg| \le \|f\|_p \|g\|_{p'}$$

□

Schwartz inequality $p = 2$

$$|\langle f, g\rangle_2| = \left|\int_\Omega f\overline{g}\right| \le \int_\Omega |fg| = \|fg\|_1 \le \|f\|_2 \|g\|_2 .$$

The L_p spaces not comparable on unbounded domains

Example We'll use $\Omega = \mathbb{R}$. Assume $0 < q < p < \infty$
Choose

$$f(x) := \begin{cases} 0 & |x| \le 1 \\ \dfrac{1}{|x|^{1/q}} & |x| > 1 \end{cases}$$

We have $f \in L_p(\mathbb{R})$, $f \notin L_q(\mathbb{R})$
Now choose

$$f(x) := \begin{cases} \dfrac{1}{|x|^{1/p}} & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

We have $f \in L_q(\mathbb{R})$, $f \notin L_p(\mathbb{R})$

Theorem If $|\Omega| < \infty$, $0 < q < p$, $f \in L_p(\Omega)$ then

$$\|f\|_{L_q(\Omega)} \le |\Omega|^{1/q-1/p} \|f\|_{L_p(\Omega)} .$$

Proof Define $r := p/q \ge 1$

$$\|f\|_q^q = \int_\Omega |f|^q = \int_\Omega |f|^q 1 \underset{Holder}{\le} \left(\int_\Omega \left(|f|^q\right)^r \right)^{1/r} \left(\int_\Omega 1^{r'} \right)^{1/r'}$$
$$= \left(\int_\Omega |f|^p \right)^{q/p} |\Omega|^{1-q/p}$$

□

Theorem Minkowski for Lp spaces $1 \le p \le \infty$, $f, g \in L_p$,

$$\|f + g\|_p \le \|f\|_p + \|g\|_p \; .$$

Proof for $1 < p < \infty$ ($p = 1, \infty$ is easier). W.l.g $f, g \ge 0$. We apply Hölder twice,

$$\begin{aligned}\int (f+g)^p &= \int f(f+g)^{p-1} + \int g(f+g)^{p-1} \\ &\le \left(\|f\|_p + \|g\|_p\right)\left(\int (f+g)^{(p-1)p'}\right)^{1/p'} \\ &= \left(\|f\|_p + \|g\|_p\right)\left(\int (f+g)^p\right)^{1-1/p} \\ &= \left(\|f\|_p + \|g\|_p\right)\left(\int (f+g)^p\right)\underbrace{\left(\int (f+g)^p\right)^{-1/p}}_{\|f+g\|_p^{-1}}.\end{aligned}$$

□

Thm For $0 < p < 1$, we have

(i) $\left\|\sum_k f_k\right\|_p^p \le \sum_k \|f_k\|_p^p$

(ii) $\|f+g\|_p \le 2^{1/p-1}\left(\|f\|_p + \|g\|_p\right)$ or in general $\left\|\sum_{k=1}^N f_k\right\|_p \le N^{1/p-1}\sum_{j=1}^N \|f_k\|_p$

Proof The quasi-triangle inequality (ii) is derived from (i), by using $1 \le p^{-1} < \infty$,

$$\left\|\sum_{k=1}^N f_k\right\|_p \underset{(i)}{\le} \left(\sum_{k=1}^N \|f_k\|_p^p\right)^{1/p} = \left(\sum_{j=1}^N 1\cdot\|f_k\|_p^p\right)^{1/p} \underset{\substack{\text{Discrete}\\ \text{Holder}}}{\le} \underbrace{\left(\sum_{k=1}^N 1^{\frac{1}{1-p}}\right)}_{N}^{(1-p)1/p} \left(\sum_{k=1}^N \|f_k\|_p\right) = N^{1/p-1}\sum_{k=1}^N \|f_k\|_p$$

To prove (i), we need the following lemma

Lemma I For $0 < p \le 1$ and any sequence of non-negative $a = \{a_k\}$,

$$\left(\sum_k a_k\right)^p \le \sum_k a_k^p$$

Proof We first prove $(a_1 + a_2)^p \le a_1^p + a_2^p$ and then apply induction.

To prove the inequality use $h(t) := t^p + 1 - (t+1)^p$. $h(0) = 0$ and $h'(t) = pt^{p-1} - p(t+1)^{p-1} \ge 0$. Therefore, $h(t) \ge 0$, for $t \ge 0$. This gives $t^p + 1 \ge (t+1)^p$. Setting $t = a_1 / a_2$ gives

$$\left(\frac{a_1}{a_2}\right)^p + 1 \ge \left(\frac{a_1}{a_2} + 1\right)^p \Rightarrow a_1^p + a_2^p \ge (a_1 + a_2)^p.$$

□

Proof of Theorem (i) : Simply apply the lemma pointwise for $x \in \Omega$

$$\left\|\sum_k f_k\right\|_p^p \le \int_\Omega \left(\sum_k |f_k(x)|\right)^p dx \le \int_\Omega \left(\sum_k |f_k(x)|^p\right) dx = \sum_k \int_\Omega |f_k(x)|^p dx = \sum_k \|f_k\|_p^p.$$

□

Definition The space $l_p(\mathbb{Z})$, $0 < p \le \infty$, is the space of sequences $a = \{a_k\}_{k\in\mathbb{Z}}$, for which the norm is finite

$$\|a\|_{l_p} := \begin{cases} \left(\sum_k |a_k|^p \right)^{1/p}, & 0 < p < \infty, \\ \sup_k |a_k|, & p = \infty. \end{cases}$$

Lemma II $l_p \subset l_q$ for $p \le q$. That is, for any sequence $a = \{a_k\}$

$$\|a\|_{l_q} \le \|a\|_{l_p} .$$

Proof Case of $q = \infty$, for any $j \in \mathbb{Z}$,

$$|a_j| = \left(|a_j|^p \right)^{1/p} \le \left(\sum_k |a_k|^p \right)^{1/p} = \|a\|_{l_p} .$$

Therefore,

$$\|a\|_{l_\infty} = \sup_j |a_j| \le \|a\|_{l_p} .$$

For $q < \infty$, we have

$$\left(\sum_k |a_k|^q \right)^{p/q} \le \sum_k \left(|a_k|^q \right)^{p/q} = \sum_k |a_k|^p \Rightarrow \left(\sum_k |a_k|^q \right)^{1/q} \le \left(\sum_k |a_k|^p \right)^{1/p} .$$

□

Hilbert spaces and $L_2(\Omega)$

Def Hilbert space H : Complete metric vector space induced by an inner product $\langle , \rangle : H \times H \to \mathbb{C}$.

Properties of the inner product:

i. symmetric $\langle f, g \rangle = \overline{\langle g, f \rangle}$,
ii. linear $\langle \alpha f_1 + \beta f_2, g \rangle = \alpha \langle f_1, g \rangle + \beta \langle f_2, g \rangle$,
iii. Positive definite $\langle f, f \rangle \ge 0$, with $\langle f, f \rangle = 0 \Leftrightarrow f = 0$.

The natural norm $\|f\|_H := \langle f, f \rangle^{1/2}$ satisfies

(i) Cauchy-Schwartz

$$|\langle f, g \rangle| \le \|f\|_H \|g\|_H$$

(ii) Triangle inequality

$$\|f + g\|^2 = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2 \le \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 = \left(\|f\| + \|g\| \right)^2$$

So an Hilbert space is a Banach space.

Examples

(i) $l_2(\mathbb{Z})$: $\langle \alpha, \beta \rangle_{l_2} := \sum_{i \in \mathbb{Z}} \alpha_i \overline{\beta}_i$, $\|\alpha\|_2 := \left(\sum_{i \in \mathbb{Z}} |\alpha_i|^2 \right)^{1/2}$

(ii) $L^2(\Omega)$: f, g measurable, $\langle f, g \rangle := C_\Omega \int_\Omega f(x) \overline{g(x)} dx$,

$$\|f\|_{L_2(\Omega)} = \|f\|_2 = \langle f, f \rangle^{1/2} = \left(C_\Omega \int_\Omega |f(x)|^2 dx \right)^{1/2} .$$

For $\Omega=\mathbb{R}^n, C_\Omega=1$. For $\Omega=[-\pi,\pi]^n, C_\Omega=\frac{1}{(2\pi)^n}$.

The distribution function

Definition For (a measurable) $f:\mathbb{R}^n\to\mathbb{C}$, define $d_f:[0,\infty)\to[0,\infty]$, by

$$d_f(\alpha):=\left|\left\{x\in\mathbb{R}^n : |f(x)|>\alpha\right\}\right|.$$

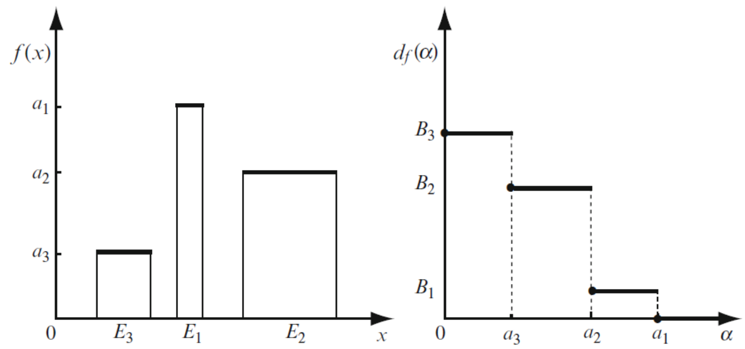


Fig. 1.1 The graph of a simple function $f=\sum_{k=1}^{3} a_k\chi_{E_k}$ and its distribution function $d_f(\alpha)$. Here $B_j=\sum_{k=1}^{j}\mu(E_k)$.

Properties

(i) If $|g(x)|\le|f(x)|$ a.e, then $d_g(\alpha)\le d_f(\alpha)$.

Easy to see since for any $\alpha>0$, if $|g(x)|>\alpha\Rightarrow|f(x)|>\alpha$, for a. e, $x\in\mathbb{R}^n$, which implies

$$d_f(\alpha)=\left|\left\{x\in\mathbb{R}^n : |f(x)|>\alpha\right\}\right|\ge\left|\left\{x\in\mathbb{R}^n : |g(x)|>\alpha\right\}\right|=d_g(\alpha).$$

(ii) $d_{cf}(\alpha)=d_f(\alpha/|c|)$, $\forall c\ne 0$ (assignment)

(iii) $d_{f+g}(\alpha+\beta)\le d_f(\alpha)+d_g(\beta)$ (assignment).

In the next theorem you may identify $X=\mathbb{R}^n$, $d\mu=dx$.

Theorem For $f\in L_p(X,d\mu)$, $0<p<\infty$,

$$\|f\|_p^p=\int_X|f|^p\,d\mu=p\int_0^\infty\alpha^{p-1}d_f(\alpha)\,d\alpha.$$

Proof

$$\begin{aligned}p\int_0^\infty\alpha^{p-1}d_f(\alpha)\,d\alpha&=p\int_0^\infty\alpha^{p-1}\left(\int_X\mathbf{1}_{\{x:|f(x)|>\alpha\}}(x)\,d\mu(x)\right)d\alpha\\&=\int_X\left(\int_0^{|f(x)|}p\alpha^{p-1}d\alpha\right)d\mu(x)\\&=\int_X|f(x)|^p\,d\mu(x)\\&=\|f\|_{L_p(X)}^p\end{aligned}$$

□

Weak L_p spaces

Definition For $0 < p < \infty$, $L_{p,\infty}$, the weak L_p space is defined as the set of all measurable functions $f : X \to \mathbb{C}$, such that there exists $0 < M < \infty$, such that

$$d_f(\alpha)^{1/p}\alpha \le M\,,\ \forall \alpha > 0\,. \qquad (*)$$

We also define $\|f\|_{L_{p,\infty}} := \inf_{M \text{ satisfies } *} M = \sup_{\alpha>0} d_f(\alpha)^{1/p}\alpha$. For $p = \infty$, we define $L_{\infty,\infty} := L_\infty$.

Example $X = \mathbb{R}^n$, $f(x) = |x|^{-n/p}$, $p < \infty$. It is easy to see $f \notin L_p$. However,

$$\begin{aligned} d_f(\alpha) &= \left|\left\{x \in \mathbb{R}^n : |x|^{-n/p} > \alpha\right\}\right| \\ &= \left|\left\{x \in \mathbb{R}^n : |x| < \alpha^{-p/n}\right\}\right| \\ &= |B(0,1)|\alpha^{-p}, \end{aligned}$$

where $B(x,r) := \{y \in \mathbb{R}^n : |y - x| < r\}$. This gives that

$$\|f\|_{L_{p,\infty}} = \sup_\alpha d_\alpha(f)^{1/p}\alpha = |B(0,1)|^{1/p} < \infty\,.$$

Theorem For $0 < p < \infty$, $L_{p,\infty}$ is a quasi-Banach space

Proof Let $f, g \in L_{p,\infty}$. By the properties of the distribution function, for $1 \le p < \infty$,

$$\begin{aligned} \|f+g\|_{L_{p,\infty}} &= \sup_{\alpha>0} d_{f+g}(2\alpha)^{1/p}(2\alpha) \\ &\underset{(iii)}{\le} 2\sup_{\alpha>0}\left(d_f(\alpha) + d_g(\alpha)\right)^{1/p}\alpha \\ &\le 2\sup_{\alpha>0}\left(d_f(\alpha)^{1/p}\alpha + d_g(\alpha)^{1/p}\alpha\right) \\ &\le 2\left(\sup_{\alpha>0} d_f(\alpha)^{1/p}\alpha + \sup_{\beta>0} d_g(\beta)^{1/p}\beta\right) \\ &\le 2\left(\|f\|_{L_{p,\infty}} + \|g\|_{L_{p,\infty}}\right). \end{aligned}$$

For $0 < p < 1$, with the same method, we get $\|f+g\|_{L_{p,\infty}} \le 2^{1/p}\left(\|f\|_{L_{p,\infty}} + \|g\|_{L_{p,\infty}}\right)$.

□

Theorem For $0 < p < \infty$, $\|f\|_{p,\infty} \le \|f\|_p$, and hence $L_p \subset L_{p,\infty}$

Proof The proof is a direct consequence of the Chebyshev inequality. Let $f \in L_p$. For any $\alpha > 0$

$$\begin{aligned} d_f(\alpha)\alpha^p &= \left|\left\{x \in \mathbb{R}^n : |f(x)| > \alpha\right\}\right|\alpha^p \\ &\le \int_{\{x\in\mathbb{R}^n : |f(x)|>\alpha\}} |f(x)|^p\,dx \\ &\le \int_{\mathbb{R}^n} |f(x)|^p\,dx = \|f\|_p^p\,. \end{aligned}$$

□

First Glimpse into interpolation (of function spaces)

Theorem Let $0 < p < q \le \infty$, and $f \in L_{p,\infty} \bigcap L_{q,\infty}$. Then, $f \in L_r$, for all $p < r < q$ and

$$\|f\|_r \le \left(\frac{r}{r-p} + \frac{r}{q-r} \right)^{1/r} \|f\|_{p,\infty}^{\frac{1/r-1/q}{1/p-1/q}} \|f\|_{q,\infty}^{\frac{1/p-1/r}{1/p-1/q}}.$$

This implies that if $f \in L_p \bigcap L_q$, then

$$\|f\|_r \le \left(\frac{r}{r-p} + \frac{r}{q-r} \right)^{1/r} \|f\|_{p}^{\frac{1/r-1/q}{1/p-1/q}} \|f\|_{q}^{\frac{1/p-1/r}{1/p-1/q}}.$$

Proof The case $q = \infty$ is easier. Recall $L_{\infty,\infty} := L_\infty$. So, we need to prove

$$\|f\|_r \le \left(\frac{r}{r-p} \right)^{1/r} \|f\|_{p,\infty}^{\frac{1/r}{1/p}} \|f\|_{\infty}^{\frac{1/p-1/r}{1/p}}.$$

Since $d_f(\alpha) = 0$ for $\alpha > \|f\|_\infty$,

$$\begin{aligned}
\|f\|_r^r &= r\int_0^{\|f\|_\infty} \alpha^{r-1} d_f(\alpha)\, d\alpha \\
&\le r\int_0^{\|f\|_\infty} \alpha^{r-1} \frac{\|f\|_{p,\infty}^p}{\alpha^p} d\alpha \\
&= r\int_0^{\|f\|_\infty} \alpha^{r-1-p} \|f\|_{p,\infty}^p\, d\alpha \\
&= \frac{r}{r-p} \|f\|_{p,\infty}^p \|f\|_\infty^{r-p}.
\end{aligned}$$

Let $0 < q < \infty$. We know that

$$d_f(\alpha) \le \min\left(\frac{\|f\|_{p,\infty}^p}{\alpha^p}, \frac{\|f\|_{q,\infty}^q}{\alpha^q} \right).$$

For

$$B := \left(\frac{\|f\|_{q,\infty}^q}{\|f\|_{p,\infty}^p} \right)^{\frac{1}{q-p}},$$

we have that

$$\alpha \le B \Leftrightarrow \alpha \le \left(\frac{\|f\|_{q,\infty}^q}{\|f\|_{p,\infty}^p} \right)^{\frac{1}{q-p}} \Leftrightarrow \alpha^{q-p} \le \frac{\|f\|_{q,\infty}^q}{\|f\|_{p,\infty}^p} \Leftrightarrow \frac{\|f\|_{p,\infty}^p}{\alpha^p} \le \frac{\|f\|_{q,\infty}^q}{\alpha^q}.$$

Then

$$
\begin{aligned}
\|f\|_r^r &= r\int_0^\infty \alpha^{r-1} d_f(\alpha)\, d\alpha \\
&\le r\int_0^\infty \alpha^{r-1} \min\left(\frac{\|f\|_{p,\infty}^p}{\alpha^p}, \frac{\|f\|_{q,\infty}^q}{\alpha^q}\right) d\alpha \\
&= r\int_0^B \alpha^{r-1-p} \|f\|_{p,\infty}^p \, d\alpha + r\int_B^\infty \alpha^{r-1-q} \|f\|_{q,\infty}^q \, d\alpha \\
&\underset{r-p>0, r-q<0}{=} \frac{r}{r-p}\|f\|_{p,\infty}^p B^{r-p} + \frac{r}{q-r}\|f\|_{q,\infty}^q B^{r-q} \\
&= \left(\frac{r}{r-p} + \frac{r}{q-r}\right)\left(\|f\|_{p,\infty}^p\right)^{\frac{q-r}{q-p}} \left(\|f\|_{q,\infty}^q\right)^{\frac{r-p}{q-p}} .
\end{aligned}
$$

This gives

$$
\begin{aligned}
\|f\|_r &\le \left(\frac{r}{r-p} + \frac{r}{q-r}\right)^{1/r} \|f\|_{p,\infty}^{\frac{p(q-r)}{r(q-p)}} \|f\|_{q,\infty}^{\frac{q(r-p)}{r(q-p)}} \\
&= \left(\frac{r}{r-p} + \frac{r}{q-r}\right)^{1/r} \|f\|_{p,\infty}^{\frac{1/r-1/q}{1/p-1/q}} \|f\|_{q,\infty}^{\frac{1/p-1/r}{1/p-1/q}} .
\end{aligned}
$$

□

First Glimpse into Hardy spaces

$\Omega = \mathbb{R}^n$, Laplace operator

$$L = -\Delta := -\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} .$$

The Heat equation $u(x,t)$

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u, \\
u(x,0) = f(x).
\end{cases}
$$

The Gaussian (heat) Kernels satisfy the Heat equation

$$\varphi_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \qquad \int_{\mathbb{R}^n} \varphi_t(x)\, dx = 1, \qquad t > 0 .$$

Convolution $f, g \in L_1(\mathbb{R}^n)$, $f * g(x) := \int_{\mathbb{R}^n} f(x-y) g(y)\, dy$.

(i) By change of variables, it is easy to see $f * g = g * f$.

(ii) $f, g \in L_1(\mathbb{R}^n) \Rightarrow f * g \in L_1(\mathbb{R}^n)$

Semi-group $\varphi_t * \varphi_s = \varphi_{t+s}$, $t, s > 0$.

Theorem If f is continuous and bounded then

$$u(x,t) = \varphi_t * f(x) = \int_{\mathbb{R}^n} f(y) \varphi_t(x-y)\, dy ,$$

solves the Heat equation with initial conditions f .

Sketch Easy to see

$$\left(\frac{\partial}{\partial t}-\Delta\right)u(x,t)=\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial t}-\Delta\right)\varphi_t(x-y)f(y)dy=0.$$

$$u(x,t)=\varphi_t * f(x)\underset{t\to 0}{\to} f(x).$$

Maximal function

$$M_\varphi f(x):=\sup_{t>0}|u(x,t)|=\sup_{t>0}|\varphi_t * f(x)|, \qquad \forall x\in\mathbb{R}^n.$$

Typical questions

(i) If we know that $f\in L_p(\mathbb{R}^n)$, $1\le p\le\infty$, what can we say about the solution? In other words, can we bound $\|M_\varphi f\|_p$?

(ii) If f is a functional acting on 'smooth' functions, then $\varphi_t * f(x):=\langle f,\varphi_t(x-\cdot)\rangle$ is well defined for every $x\in\mathbb{R}^n$. When can we say something about $M_\varphi f$?

The Schwartz Class

Let $\varphi:\mathbb{R}^n\to\mathbb{C}$. A partial derivative of order m

$$\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}_+^n, \quad \partial^\alpha\varphi:=\frac{\partial^m\varphi}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}, \quad |\alpha|:=\sum_{i=1}^n\alpha_i=m.$$

Properties of multivariate monomials:

(i) $x^\alpha:=\prod_{i=1}^n x_i^{\alpha_i}$, $\qquad x\in\mathbb{R}^n$, $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}_+^n$.

(ii) For any $\alpha\in\mathbb{Z}_+^n$, $|x^\alpha|\le C(n,\alpha)|x|^{|\alpha|}$, where $|x|:=\sqrt{\sum_{i=1}^n x_i^2}$.

Proof Define $\phi(x):=x^\alpha$. This function is continuous and so let $C(n,\alpha):=\|\phi\|_{L_\infty\left(\overline{B(0,1)}\right)}<\infty$, where $B(0,1):=\{x\in\mathbb{R}^n : |x|<1\}$. Thus, $|x^\alpha|\le C(n,\alpha)$, $\forall x\in\overline{B(0,1)}$. For arbitrary $0\ne x\in\mathbb{R}^n$,

$$\frac{x}{|x|}\in\overline{B(0,1)}\Rightarrow\left|\left(\frac{x}{|x|}\right)^\alpha\right|\le C\Rightarrow\frac{|x^\alpha|}{|x|^{|\alpha|}}\le C\Rightarrow|x^\alpha|\le C|x|^{|\alpha|}.$$

□

(iii) [Inverse] For any $k\ge 1$, $\exists C(n,k)>0$, such that

$$|x|^k\le C(n,k)\sum_{|\alpha|=k}|x^\alpha|.$$

Proof We claim $C(n,k)^{-1}:=\min_{|x|=1}\sum_{|\alpha|=k}|x^\alpha|>0$. Several ways to see this.

a. For any $x\in\mathbb{R}^n$, $|x|=1$

$$\sum_{|\alpha|=k}|x^\alpha|\ge\sum_{i=1}^n|x_i|^k>0.$$

If $\inf_{|x|=1}\sum_{|\alpha|=k}|x^\alpha|=0$, then since $\mathbb{S}^{n-1}:=\{x\in\mathbb{R}^n:|x|=1\}$ is compact, we arrive at a contradiction by

$$\inf_{|x|=1}\sum_{|\alpha|=k}|x^\alpha|=0\Rightarrow\min_{|x|=1}\sum_{|\alpha|=k}|x^\alpha|=0\Rightarrow\exists x\in\mathbb{R}^n,|x|=1,\ \sum_{i=1}^n|x_i|=0.$$

b. Similar approach: use the equivalence of finite dimensional Banach spaces $l_k(n) \sim l_2(n)$:

$$\sum_{|\alpha|=k} \left| x^{\alpha} \right| \geq \sum_{i=1}^{n} |x_i|^k = |x|_k^k \geq C(n,k)^{-1} |x|_2^k = C(n,k)^{-1}.$$

Then, $\forall x \in \mathbb{R}^n \setminus \{0\}$

$$\sum_{|\alpha|=k} \left| \left(\frac{x}{|x|} \right)^{\alpha} \right| \geq C(n,k)^{-1} \Rightarrow |x|^k \leq C(n,k) \sum_{|\alpha|=k} \left| x^{\alpha} \right|$$

□

Definition $C^m(\Omega)$: The space of all continuously differentiable functions of order m in the classical sense.

$$\|\varphi\|_{C^m(\Omega)} := \sum_{|\alpha| \leq m} \left\| \partial^{\alpha} \varphi \right\|_{L_\infty(\Omega)} < \infty.$$

The ***semi-norm*** (there could be elements $f \in C^m(\Omega)$, $|f| = 0$, $f \neq 0$)

$$|\varphi|_{C^m(\Omega)} := \sum_{|\alpha|=m} \left\| \partial^{\alpha} \varphi \right\|_{\infty}.$$

Example $C^m[a,b]$ Then $\|\varphi\|_{C^m[a,b]} = \sum_{k=0}^{m} \left\| \varphi^{(k)} \right\|_{\infty}$ is a norm, $|\varphi|_{C^m[a,b]} = \left\| \varphi^{(m)} \right\|_{L_\infty[a,b]}$ is a semi-norm with the polynomials of degree $m-1$ as a null-space.

Def The Schwartz class S, is the set of C^{∞} functions $\varphi : \mathbb{R}^n \to \mathbb{C}$, such that for all $\alpha, \beta \in \mathbb{Z}_+^n$, there exists $C_{\varphi}(\alpha, \beta) > 0$, such that

$$\sup_{x \in \mathbb{R}^n} \left| x^{\alpha} \partial^{\beta} \varphi(x) \right| \leq C_{\varphi}(\alpha, \beta).$$

The set $\{C_{\varphi}(\alpha, \beta)\}$ is called the set of Schwartz semi-norms of φ.

Examples/properties

(i) $e^{-|x|^2} \in S$ because it is in C^{∞} and decays faster than any polynomial.

(ii) $e^{-|x|} \notin S$, it is not C^{∞}.

(iii) $C_0^{\infty}(\mathbb{R}^n) \subset S(\mathbb{R}^n)$ (compactly supported smooth functions)

(iv) Alternative definition (assignment). For any $\alpha \in \mathbb{Z}_+^n$, and $N > 0$, $\exists C_{\varphi}(\alpha, N) > 0$ such that

$$\left| \partial^{\alpha} \varphi(x) \right| \leq C_{\varphi}(\alpha, N) (1 + |x|)^{-N}.$$

(v) $\varphi_k \underset{S}{\to} \varphi$, as $k \to \infty$, if $\sup_{x \in \mathbb{R}^n} \left| x^{\alpha} \partial^{\beta} (\varphi - \varphi_k)(x) \right| \to 0$, $\forall \alpha, \beta \in \mathbb{Z}_+^0$.

(vi) $\varphi \in S \Rightarrow \partial^{\alpha} \varphi \in L_p$, for all $\alpha \in \mathbb{Z}_+^n$, $0 < p \leq \infty$.

Proof For $0 < p < \infty$, let $N > (n+1)/p$.

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \partial^{\alpha} \varphi(x) \right|^p dx &= \int_{\mathbb{R}^n} \left| (1+|x|)^{(n+1)/p} \partial^{\alpha} \varphi(x) \right|^p (1+|x|)^{-(n+1)} dx \\ &\leq \sup_{x \in \mathbb{R}^n} \left| (1+|x|)^N \partial^{\alpha} \varphi(x) \right|^p \int_{\mathbb{R}^n} (1+|x|)^{-(n+1)} dx \\ &\leq C(\alpha, N)^p C(n) \end{aligned}$$

□

Tempered Distributions

Definition The dual space the space of continuous linear functionals. The dual space of $C_0^\infty\left(\mathbb{R}^n\right)$ denoted by $\left(C_0^\infty\left(\mathbb{R}^n\right)\right)' := D'\left(\mathbb{R}^n\right)$, is the space of **distributions**. The dual space $S'\left(\mathbb{R}^n\right)$ is the space of **tempered distributions**. We will denote the action $f \in S'$ on $\varphi \in S$, by $\langle f, \varphi \rangle$. This means that if $\varphi_k \underset{S}{\to} \varphi$, as $k \to \infty$, then $\langle f, \varphi_k \rangle \underset{k\to\infty}{\to} \langle f, \varphi \rangle$. We will assume the following stronger assumption: A linear functional f is in S' iff there exist $C > 0$, m, k such that

$$\left|\langle f, \varphi \rangle\right| \le C \sum_{|\alpha|\le m, |\beta|\le k} C_\varphi(\alpha, \beta), \qquad \forall \varphi \in S.$$

Observe that if this is satisfied for a linear functional f, then for any $\varphi_j \underset{S}{\to} \varphi$,

$$\begin{aligned}\left|\langle f, \varphi \rangle - \langle f, \varphi_j \rangle\right| &= \left|\langle f, \varphi - \varphi_j \rangle\right| \\ &\le C \sum_{|\alpha|\le m, |\beta|\le k} C_{\varphi-\varphi_j}(\alpha, \beta) \underset{j\to\infty}{\to} 0.\end{aligned}$$

Examples

(i) We already discussed the Dirac $\langle \delta_{x_0}, \varphi \rangle := \varphi(x_0)$, $x_0 \in \mathbb{R}^n$.

(ii) If $f \in L_p\left(\mathbb{R}^n\right)$, $1 \le p \le \infty$, then since $S \subset L_{p'}\left(\mathbb{R}^n\right)$, $\frac{1}{p} + \frac{1}{p'} = 1$, we can define $\langle f, \varphi \rangle := \int_{\mathbb{R}^n} f\varphi$ and by Hölder, for $N > (n+1)/p'$

$$\begin{aligned}\left|\langle f, \varphi \rangle\right| &\le \|f\|_p \|\varphi\|_{p'} \\ &\le C\|f\|_p C_\varphi(0, N).\end{aligned}$$

(iii) Algebraic polynomials – Let $P \in \Pi_{r-1}\left(\mathbb{R}^n\right)$, $P(x) = \sum_{|\alpha|<r} a_\alpha x^\alpha$, $a_\alpha \in \mathbb{C}$. Then one can define $\langle P, \varphi \rangle := \int_{\mathbb{R}^n} P\varphi$. Easy to see that

$$\begin{aligned}\left|\langle P, \varphi \rangle\right| &\le \sum_{|\alpha|<r} |a_\alpha| \int_{\mathbb{R}^n} \left|x^\alpha\right| \left|\varphi(x)\right| dx \\ &\le C(n, r) \max_{|\alpha|<r} |a_\alpha| \sum_{|\alpha|<r} C_\varphi\left(0, |\alpha| + n + 1\right) \int_{\mathbb{R}^n} |x|^{|\alpha|} \left(1 + |x|\right)^{-(|\alpha|+n+1)} dx \\ &\le C(n, r) \max_{|\alpha|<r} |a_\alpha| C_\varphi(0, r+n)\end{aligned}$$

(iv) Any function that satisfies $\left|f(x)\right| \le C\left(1 + |x|\right)^M$, for some $M > 0$. Same proof as for polynomials (polynomial growth).

Definition Distributional (generalized) derivative of $f \in S'$. Let $\alpha \in \mathbb{Z}_+^n$. Then

$$\langle \partial^\alpha f, \varphi \rangle := (-1)^{|\alpha|} \langle f, \partial^\alpha \varphi \rangle.$$

This definition is in line with integration by parts. If $\varphi, \psi \in S$, then

$$\int_{\mathbb{R}^n} \left(\partial^\alpha \varphi\right)\psi = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \varphi\left(\partial^\alpha \psi\right).$$

Remarks:

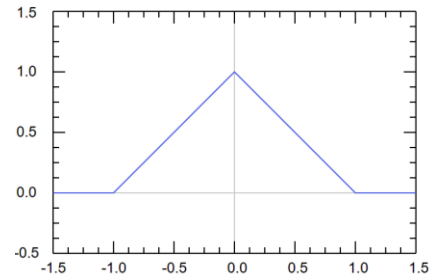
(i) Notice we are using $\left|\partial^\beta \varphi(x)\right|, \left|\partial^\beta \psi(x)\right| \underset{|x|\to\infty}{\to} 0$, $\forall \beta \in \mathbb{Z}_+^n$ and $\varphi, \psi \in S$.

(ii) We will later require that the distributions and their distributional derivatives are functions.

Examples

(i) $\langle \partial^\alpha \delta_{x_0}, \varphi \rangle := (-1)^{|\alpha|} \langle \delta_{x_0}, \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(x_0)$.

(ii) **Assignment:** For $H(x) := \begin{cases} x+1, & -1 \le x < 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & else. \end{cases}$



prove that $H'(x) = g(x) = \begin{cases} 1, & -1 \le x < 0, \\ -1, & 0 \le x \le 1, \\ 0, & else. \end{cases}$

Examples for other operations on distributions via duality:

(i) Composition with an invertible matrix. What is $f(M\cdot)$, for $f \in S'$? If f is a function, then

$$\int_{\mathbb{R}^n} f(Mx)\varphi(x)dx \underset{y=Mx}{=} \left|\det M^{-1}\right| \int_{\mathbb{R}^n} f(y)\varphi(M^{-1}y)dy.$$

So we define $\langle f(M\cdot), \varphi \rangle := \langle f, \left|\det M^{-1}\right| \varphi(M^{-1}\cdot) \rangle$, $\forall \varphi \in S$.

(ii) What is a compactly supported distribution? Again, we define by duality. We say $\mathrm{supp}(f) = \Omega$, if $\langle f, \varphi \rangle = 0$, for any $\varphi \in S$, with $\mathrm{supp}(\varphi) \subseteq \Omega^c$.

Sobolev spaces

Definition We define the space of ***test-functions*** $C_0^r(\Omega)$ - continuously r-differentiable with compact support in Ω.

Definition Sobolev spaces $W_p^r(\Omega)$, $1 \le p \le \infty$

Def I For $1 \le p < \infty$, completion of $C_0^r(\Omega)$ with respect to the norm $\sum_{|\alpha| \le r} \left\| \partial^\alpha f \right\|_{L_p(\Omega)}$. For $p = \infty$, we take $W_\infty^r(\Omega) := C^r(\Omega)$.

Def II Let $f \in L_p(\Omega)$. Now for $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \le r$, $g := \partial^\alpha f$ is the ***distributional (generalized) derivative*** of f if it is a function and for all $\phi \in C_0^r(\Omega)$

$$\int_\Omega g\phi = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha \phi.$$

So, in this sense $H \in W_p^1(\mathbb{R})$, $1 \le p < \infty$.

Assignment: Use cubic Hermite interpolation to find a sequence of functions $\{f_k\} \subset C^1[-1,1]$, such that $f_k \to H$ in W_1^1. Hint: Create Hermite cubic polynomials over $[-1/k, 1/k]$, $k \ge 1$.

The Sobolev norm and semi-norm. We require that the distributional derivatives exist as functions(!) in $L_p(\Omega)$ and

$$\|f\|_{W_p^r(\Omega)} := \sum_{|\alpha|\le r} \left\|\partial^\alpha f\right\|_{L_p(\Omega)} < \infty \qquad\qquad |f|_{W_p^r(\Omega)} := \sum_{|\alpha|=r} \left\|\partial^\alpha f\right\|_{L_p(\Omega)} .$$

Theorem $W_p^r(\Omega)$ is a Banach space

Proof We only need to prove completeness. Let $\{f_k\}$ be a Cauchy sequence in W_p^r. Then $\partial^\alpha f_k$ is a Cauchy sequence in L_p, $\forall \alpha, |\alpha| \le r$. Since L_p is complete, there exist $f_\alpha \in L_p$ as the limits. We can view them as distributions. Let $\varphi \in S$, then

$$\left|\left\langle \partial^\alpha f_k, \varphi\right\rangle - \left\langle f_\alpha, \varphi\right\rangle\right| \le \int_\Omega \left|\partial^\alpha f_k - f_\alpha\right| |\varphi| \le \left\|\partial^\alpha f_k - f_\alpha\right\|_p \|\varphi\|_{p'}$$

Therefore $\partial^\alpha f_k \underset{S'}{\to} f_\alpha$. This means that

$$\left\langle f_\alpha, \varphi\right\rangle = \lim_{k\to\infty}\left\langle \partial^\alpha f_k, \varphi\right\rangle = \lim_{k\to\infty}(-1)^{|\alpha|}\left\langle f_k, \partial^\alpha\varphi\right\rangle = (-1)^{|\alpha|}\left\langle f, \partial^\alpha\varphi\right\rangle .$$

Therefore f_α are the distributional derivatives of f and so f, the limit, is in W_p^r.

□

Theorem For a 'smooth' domain $\Omega \subseteq \mathbb{R}^n$, $1 \le p < \infty$ and $r \ge 1$, there exists a constant $C > 0$, such that for any $0 < j < r$, $\varepsilon > 0$ and $f \in W_p^r(\Omega)$,

$$|f|_{j,p} \le C\left(\varepsilon |f|_{r,p} + \varepsilon^{-j/(r-j)} \|f\|_p\right) .$$

Remarks

(i) Sometimes one sees in text books a definition $\|f\|_{W_p^r(\Omega)} := \|f\|_{L_p(\Omega)} + |f|_{W_p^r(\Omega)}$, since by the theorem with $\varepsilon = 1$

$$\|f\|_{L_p(\Omega)} + |f|_{W_p^r(\Omega)} \le \sum_{|\alpha|\le r}\left\|\partial^\alpha f\right\|_{L_p(\Omega)} \le C\left(\|f\|_{L_p(\Omega)} + |f|_{W_p^r(\Omega)}\right).$$

(ii) The constant C depends on the 'smoothness' of the boundary $\partial\Omega$.

We will prove the theorem for $\Omega = \mathbb{R}^n$. First, we need two lemmas

Lemma If $g \in C^2[0,\delta]$ and $1 \le p < \infty$, then

$$|g'(0)| \le \frac{C(p)}{\delta}\left(\delta^p \int_0^\delta |g''(u)|^p\, du + \delta^{-p}\int_0^\delta |g(u)|^p\, du\right)$$

Proof Let $g \in C^2[0,1]$ and $x \in [0,1/3], y \in [2/3,1]$. By the mean value theorem, there exists $z \in (x,y)$

$$|g'(z)| = \frac{|g(y)-g(x)|}{|y-x|} \le 3\left(|g(x)| + |g(y)|\right).$$

Therefore,

$$|g'(0)| = \left|g'(z) - \int_0^z g''(u)\,du\right| \le 3\left(|g(x)| + |g(y)|\right) + \int_0^1 |g''(u)|\,du .$$

Integration $\int_0^{1/3}\int_{2/3}^1 dxdy$ on both sides gives

$$\frac{1}{9}|g'(0)| \le \int_0^{1/3} |g(x)|\,dx + \int_{2/3}^1 |g(y)|\,dy + \frac{1}{9}\int_0^1 |g''(u)|\,du$$

So for $p=1$ we have

$$\left|g'(0)\right| \le 9\left(\int_0^1 \left|g(u)\right| du + \int_0^1 \left|g''(u)\right| du\right)$$

For $1 < p < \infty$ we get by Hölder

$$\left|g'(0)\right| \le 9\left(\left(\int_0^1 \left|g(u)\right|^p du\right)^{1/p} + \left(\int_0^1 \left|g''(u)\right|^p du\right)^{1/p}\right),$$

Which gives

$$\left|g'(0)\right|^p \le c(p)\left(\int_0^1 \left|g(u)\right|^p du + \int_0^1 \left|g''(u)\right|^p du\right).$$

Now for $\delta > 0$ define $\tilde{g}(u) := g(\delta u),\ 0 \le u \le 1.$ $\quad \tilde{g}'(u) := \delta g'(\delta u) \Rightarrow g'(0) = \delta^{-1}\tilde{g}'(0)$

$$\begin{aligned}\left|g'(0)\right|^p &= \delta^{-p}\left|\tilde{g}'(0)\right|^p = C\delta^{-p}\left(\int_0^1 \left|\tilde{g}(u)\right|^p du + \int_0^1 \left|\tilde{g}''(u)\right|^p du\right)\\ &= C\delta^{-p}\left(\delta^{-1}\int_0^\delta \left|g(v)\right|^p dv + \delta^{2p-1}\int_0^\delta \left|g''(v)\right|^p dv\right)\\ &= C\delta^{-1}\left(\delta^{-p}\int_0^\delta \left|g(v)\right|^p dv + \delta^{p}\int_0^\delta \left|g''(v)\right|^p dv\right)\end{aligned}$$

□

Lemma For $1 \le p < \infty$, $\delta > 0$ and $f \in W_p^2\left(\mathbb{R}^n\right)$

$$|f|_{1,p} \le C\left(\delta |f|_{2,p} + \delta^{-1}\|f\|_p\right)$$

Proof By density, sufficient to prove for $f \in C^2\left(\mathbb{R}^n\right) \cap W_p^2\left(\mathbb{R}^n\right)$. For any $x \in \mathbb{R}^n$ and $1 \le j \le n$, denoting $\vec{u}_j := \left(0,\ldots,0,u_j,0,\ldots,0\right)$, and applying previous lemma

$$\left|\frac{\partial f}{\partial x_j}(x)\right|^p \le C\delta^{-1}\left(\delta^{-p}\int_0^\delta \left|f\left(x+\vec{u}_j\right)\right|^p du_j + \delta^p\int_0^\delta \left|\frac{\partial^2}{\partial^2 x_j} f\left(x+\vec{u}_j\right)\right|^p du_j\right)$$

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\frac{\partial f}{\partial x_j}(x)\right|^p dx &= \int_{x_k \ne x_j}\int_{-\infty}^{\infty}\left|\frac{\partial f}{\partial x_j}(x)\right|^p dx_j\\ &\le C\delta^{-1}\int_{x_k \ne x_j}\left(\delta^{-p}\int_0^\delta\int_{-\infty}^{\infty}\left|f\left(x+\vec{u}_j\right)\right|^p dx_j du_j + \delta^p\int_0^\delta\int_{-\infty}^{\infty}\left|\frac{\partial^2}{\partial^2 x_j} f\left(x+\vec{u}_j\right)\right|^p dx_j du_j\right)\\ &\le C\int_{x_k \ne x_j}\left(\delta^{-p}\int_{-\infty}^{\infty}\left|f(x)\right|^p dx_j + \delta^p\int_{-\infty}^{\infty}\left|\frac{\partial^2}{\partial^2 x_j} f(x)\right|^p dx_j\right)\\ &\le C\left(\delta^{-p}\|f\|_p^p + \delta^p\left\|\frac{\partial^2}{\partial^2 x_j} f\right\|_p^p\right).\end{aligned}$$

This gives

$$\left\|\frac{\partial f}{\partial x_j}\right\|_p \le C\left(\delta^{-1}\|f\|_p + \delta\left\|\frac{\partial^2}{\partial^2 x_j} f\right\|_p\right) \le C\left(\delta^{-1}\|f\|_p + \delta|f|_{2,p}\right), \quad 1 \le j \le n.$$

Which implies

$$|f|_{1,p} \le C\left(\delta^{-1}\|f\|_p + \delta|f|_{2,p}\right).$$

□

Proof of theorem For $0 < j < r$, and any $\varepsilon > 0$, choose $\delta = \varepsilon^{1/(r-j)}$. The proof is by double induction on r, j. First, we prove for $j = r-1$. Assume that for some $2 \le k \le r-1$

$$|f|_{k-1,p} \le C\left(\eta|f|_{k,p} + \eta^{-(k-1)}\|f\|_p\right) .$$

Then, by the previous lemma

$$\begin{aligned}|f|_{k,p} &\le C\left(\delta|f|_{k+1,p} + \delta^{-1}|f|_{k-1,p}\right)\\ &\le C\left(\delta|f|_{k+1,p} + \delta^{-1}\eta|f|_{k,p} + \delta^{-1}\eta^{1-k}\|f\|_p\right)\end{aligned}$$

Choose η such that $C\delta^{-1}\eta = 1/2$. Then

$$|f|_{k,p} \le C\left(\delta|f|_{k+1,p} + \delta^{-k}\|f\|_p\right).$$

We now prove downward induction on j. We assume $|f|_{j,p} \le C\left(\delta^{r-j}|f|_{r,p} + \delta^{-j}\|f\|_p\right)$, for $2 \le j < r$.

$$\begin{aligned}|f|_{j-1,p} &\le C\left(\delta|f|_{j,p} + \delta^{-(j-1)}\|f\|_p\right)\\ &\le C\left(\delta\left(\delta^{r-j}|f|_{r,p} + \delta^{-j}\|f\|_p\right) + \delta^{-(j-1)}\|f\|_p\right)\\ &\le C\left(\delta^{r-(j-1)}|f|_{r,p} + \delta^{-(j-1)}\|f\|_p\right).\end{aligned}$$

Setting $\delta = \varepsilon^{1/(r-j)}$ gives

$$|f|_{j,p} \le C\left(\delta^{r-j}|f|_{r,p} + \delta^{-j}\|f\|_p\right) = C\left(\varepsilon|f|_{r,p} + \varepsilon^{-j/(r-j)}\|f\|_p\right)$$

□

Application of Sobolev space: Approximation with the Fourier Series

We now focus on the domain $\mathbb{T}^n = [-\pi,\pi]^n$ and 2π - periodic functions. They are extended to all of $\mathbb{R}^n$ by $f(x+2\pi k) = f(x)$, $k \in \mathbb{Z}^n$, $x \in \mathbb{T}^n$.

Periodic…what does it mean for us? For example, the function $f(x) = x$ is <u>not</u> continuous as a periodic function.

$L_2(\mathbb{T}^n)$ is a Hilbert space equipped with the dot-product

$$\langle f, g\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(x)\overline{g(x)}dx .$$

The exponents are an **orthonormal basis**

$$f(x) \underset{L_2}{=} \sum_{k\in\mathbb{Z}^n} \hat{f}(k)e^{ikx}, \quad \hat{f}(k) = \langle f, e^{ik\cdot}\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(x)e^{-ikx}dx, \qquad kx := \langle k, x\rangle_{l_2(\mathbb{R}^n)} = \sum_{i=1}^n k_i x_i .$$

The partial Fourier series

$$S_N f(x) := \sum_{|k|\le N} \hat{f}(k)e^{ikx} .$$

Convergence in L_2 means that for any $f \in L_2(\mathbb{T}^n)$,

$$\lim_{N\to\infty}\|f-S_N f\|_{L_2(\mathbb{T}^n)}^2=\lim_{N\to\infty}\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}|f(x)-S_N f(x)|^2\,dx=0.$$

Parseval identity

$$\|f\|_2^2=\sum_k\left|\hat f(k)\right|^2.$$

Observe **convergence is not pointwise**! There exists a continuous periodic function $f:\mathbb{T}\to\mathbb{R}$ such that

$$|S_N f(0)|=\left|\sum_{k=-N}^{N}\hat f(k)\right|\underset{N\to\infty}{\to}\infty.$$

There are even more exotic constructions! Conclusion: "Don't bring a knife to a gun fight" = Don't apply in L_∞ an Hilbert space/ L_2 tool. [This is covered in the "Approximation Theory" course].

Now let's try to say something about rate of convergence. Here is a typical approximation theoretical result: a **Jackson-type estimate**.

Theorem There exists a constant $C(r)>0$, such that for any $f\in W_2^r(\mathbb{T})$,

$$E_N(f)_2:=\|f-S_N f\|_{L_2(\mathbb{T})}\le C(r)N^{-r}|f|_{r,2},\qquad |f|_{r,2}=\left\|f^{(r)}\right\|_{L_2(\mathbb{T})}.$$

Let's prove a slightly weaker **Jackson-type estimate** (simpler proof)

Theorem Let $f\in W_2^{r+1}(\mathbb{T})$ then

$$E_N(f)_2\le C(r)N^{-(r+1/2)}|f|_{r+1,2}.$$

Proof First, assume $f\in C^{r+1}(\mathbb{T})$.

1. Decay of the Fourier coefficients -

By Parseval we have

$$\|f-S_N f\|_{L_2(\mathbb{T})}=\left\|\sum_{|k|\ge N+1}\hat f(k)e^{ikw}\right\|_2=\sqrt{\sum_{|k|\ge N+1}\left|\hat f(k)\right|^2}$$

We show $\left|\hat f(k)\right|\le|k|^{-(r+1)}\left\|f^{(r+1)}\right\|_2$, $k\neq 0$. Integration by parts yields,

$$\begin{aligned}\hat f(k)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-ikx}dx\\&=\frac{1}{2\pi}\left(\underbrace{\left.\frac{f(x)e^{-ikx}}{-ik}\right|_{-\pi}^{\pi}}_{=0}+\frac{1}{ik}\int_{-\pi}^{\pi}f'(x)e^{-ikx}dx\right)\\&=\frac{1}{ik}(f')^{\wedge}(k).\end{aligned}$$

By repeated application of the above

$$\left|\hat f(k)\right|\le|k|^{-(r+1)}\left|\left(f^{(r+1)}\right)^{\wedge}(k)\right|\le|k|^{-(r+1)}\left\|f^{(r+1)}\right\|_2.$$

Note: $f\in C^r(\mathbb{T})\Rightarrow|k|^r\left|\hat f(k)\right|=\left|\left(f^{(r)}\right)^{\wedge}(k)\right|\Rightarrow\sum_k|k|^{2r}\left|\hat f(k)\right|^2=\left\|f^{(r)}\right\|_2^2\Rightarrow\sum_k|k|^{2r}\left|\hat f(k)\right|^2<\infty$.

We shall later see that the Sobolev space W_2^r can be characterized by this 'Fourier' domain property.

2. <u>The estimate of the tail</u>

$$\left\| f - S_N f \right\|_2^2 = \sum_{|k| \ge N+1} \left| \hat{f}(k) \right|^2 \le 2 \left\| f^{(r+1)} \right\|_2^2 \sum_{k=N+1}^{\infty} \frac{1}{k^{2(r+1)}}$$

$$\sum_{k=N+1}^{\infty} \frac{1}{k^{\alpha}} \le \int_N^{\infty} \frac{dx}{x^{\alpha}} = \frac{1}{\alpha - 1} N^{-(\alpha-1)} \Rightarrow \sum_{k=N+1}^{\infty} \frac{1}{k^{2(r+1)}} \le \frac{1}{2r+1} N^{-(2r+1)}$$

$$\left\| f - S_N f \right\|_2 \le \sqrt{\frac{2}{2r+1}} N^{-(r+1/2)} \left\| f^{(r+1)} \right\|_2 .$$

3. <u>Density</u> - If $f \in W_2^{r+1}(\mathbb{T})$, we apply the density of $C^{r+1}(\mathbb{T})$ in $W_2^{r+1}(\mathbb{T})$ (assignment).

□

The origins of the Fourier Series (… which reveal how to generalize it)

The Heat equation over $\mathbb{T}$, $t \ge 0$, $u(x,t)$, $-\pi \le x \le \pi$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, \\ u(x,0) = f(x). \end{cases}$$

Laplace operator $Lf := -\Delta f = -f''$.

$$-\left(e^{ikx}\right)'' = k^2 e^{ikx} \Rightarrow \quad \left\{e^{ikx}\right\} \text{ eigenfunctions of } L,\ k^2 \text{ eigenvalues of } L.$$

$$Lf(x) = -\Delta f(x) = \sum_k k^2 \hat{f}(k) e^{ikx}, \qquad \forall f \in C^2(\mathbb{T}).$$

For $\varphi : \mathbb{R} \to \mathbb{R}$, even, define $\varphi(L) f(x) := \sum_k \varphi\left(k^2\right) \hat{f}(k) e^{ikx}$.

Spectral representation to solution of the heat equation with boundary condition f, is through **semi-group** $\varphi_t(u) := e^{-tu}$, $t > 0$. The solution is

$$u(x,t) = \varphi_t(L) f(x) = \sum_k e^{-tk^2} \hat{f}(k) e^{ikx}.$$

Example Let $\varphi(u) = 1_{[-1,1]}(u)$. Then,

$$\varphi\left(N^{-1}\sqrt{L}\right) f(x) = \sum_k \varphi\left(\frac{|k|}{N}\right) \hat{f}(k) e^{ikx} = \sum_{k=-N}^{N} \hat{f}(k) e^{ikx} = S_N f(x).$$

Fourier Transform

A <u>rigorous</u> approach is to first define the Fourier transform only for Schwartz functions.

Def $\varphi \in S\left(\mathbb{R}^n\right)$, then

$$\hat{\varphi}(w) := \int_{\mathbb{R}^n} \varphi(x) e^{-iw\cdot x} dx = \int_{\mathbb{R}^n} \varphi(x) e^{-i\langle w,x\rangle} dx, \qquad w \in \mathbb{R}^n.$$

Properties of the Fourier integral:

i. If we expand the definition to $f \in L_1\left(\mathbb{R}^n\right)$, $\left\| \hat{f} \right\|_{\infty} \le \left\| f \right\|_1$.

ii. For $f \in L_1(\mathbb{R})$, $\hat{f}$ is uniformly continuous.

Proof For $\varepsilon > 0$, let $M > 0$ such that $\int_{\mathbb{R}^n \setminus [-M,M]^n} |f| \le \varepsilon$. Then, for any $\delta \in \mathbb{R}^n$

$$\begin{aligned}
\left|\hat{f}(w+\delta) - \hat{f}(w)\right| &= \left|\int_{\mathbb{R}^n} e^{-iwx}\left(e^{-i\delta x} - 1\right) f(x)\,dx\right| \\
&\le \int_{\mathbb{R}^n} \left|e^{-i\delta x} - 1\right| \left|f(x)\right| dx \\
&= \int_{[-M,M]^n} \left|e^{-i\delta x} - 1\right| \left|f(x)\right| dx + \int_{\mathbb{R}^n \setminus [-M,M]^n} \left|e^{-i\delta x} - 1\right| \left|f(x)\right| dx \\
&\le \sup_{x \in [-M,M]^n} \left|e^{-i\delta x} - 1\right| \|f\|_1 + 2\varepsilon \underset{|\delta| \to 0}{\longrightarrow} 2\varepsilon.
\end{aligned}$$

iii. $\left(f(\cdot - z)\right)^\wedge(w) = \int_{\mathbb{R}^n} f(x-z) e^{-iwx} dx = \int_{\mathbb{R}^n} f(y) e^{-iw(y+z)} dy = e^{-iwz} \hat{f}(w), \qquad z \in \mathbb{R}^n$.

iv. For $\varphi \in S(\mathbb{R})$, $\left(\varphi^{(r)}\right)^\wedge(w) = (iw)^r \hat{\varphi}(w)$

$$\begin{aligned}
(\varphi')^\wedge(w) &= \int_{-\infty}^{\infty} \varphi'(x) e^{-iwx} dx \\
&= \underbrace{\varphi(x) e^{-iwx}\Big|_{-\infty}^{\infty}}_{=0} + iw \int_{-\infty}^{\infty} \varphi(x) e^{-iwx} dx \\
&= iw \int_{-\infty}^{\infty} \varphi(x) e^{-iwx} dx = iw\hat{\varphi}(w)
\end{aligned}$$

Examples:

i. $f(x) = 1_{[0,1]}(x)$. Then, $\hat{f}(w) = \int_0^1 e^{-iwx} dx = \left.\frac{e^{-iwx}}{-iw}\right|_0^1 = \frac{e^{-iw} - 1}{-iw} = \frac{1 - e^{-iw}}{iw}$.

ii. $f(x) = \operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$, $\hat{f}(w) = 1_{[-\pi,\pi]}(w)$. <u>Careful!</u> $\operatorname{sinc} \in L_2 \setminus L_1$.

iii. Gaussians $g_\alpha(x) = e^{-\alpha x^2}$, $\hat{g}_\alpha(w) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{w^2}{4\alpha}}$, $\alpha > 0$. This also implies a special case of the inverse Fourier

$$g_\alpha(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}_\alpha(w) e^{iwx} dw, \ \forall x \in \mathbb{R}.$$

Convolution on $\mathbb{R}^n$

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\,dy.$$

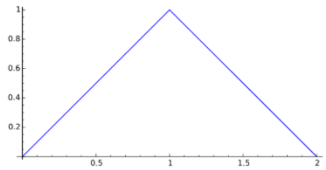
Theorem For $f, g \in L_1(\mathbb{R}^n)$, $(f * g)^\wedge(w) = \hat{f}(w)\hat{g}(w)$, $w \in \mathbb{R}^n$.

Proof

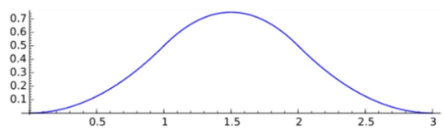
$$\begin{aligned}
(f*g)^\wedge(w) &= \int_{\mathbb{R}^n} f*g(x) e^{-iwx} dx = \int_{\mathbb{R}^n} e^{-iwx} dx \int_{\mathbb{R}^n} f(x-y) g(y)\,dy \\
&= \int_{\mathbb{R}^n} g(y) \left(\int_{\mathbb{R}^n} f(x-y) e^{-iwx} dx\right) dy \\
&= \int_{\mathbb{R}^n} g(y) e^{-iwy} \hat{f}(w)\,dy \\
&= \hat{f}(w)\hat{g}(w).
\end{aligned}$$

Examples: B-splines $N_1(x) = 1_{[0,1]}(x)$

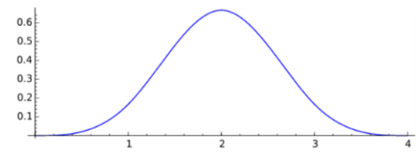
$$N_2(x) = N_1 * N_1(x) = \int_0^1 1_{[0,1]}(x-t)\,dt \underset{\substack{0\le x-t\le 1\\ x-1\le t\le x}}{=} \int_{\max(x-1,0)}^{\min(x,1)} dt = \begin{cases} 0 & x<0 \\ x & 0\le x<1 \\ 2-x & 1\le x\le 2 \\ 0 & x>2 \end{cases}$$



N_2



N_3



N_4

We define $N_r := N_{r-1} * N_1 = \underbrace{N_1 * \cdots * N_1}_{r}$. Therefore, $(N_r)^{\wedge}(w) = \left(\dfrac{1-e^{-iw}}{iw}\right)^r$.

Theorem If $\varphi \in S$, $\int \varphi = 1$, and $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, then, for $\varphi_t := t^{-n}\varphi(t^{-1}\cdot)$

$$\|f - \varphi_t * f\|_p \underset{t\to 0}{\to} 0.$$

Corollary S is dense in L_p, $1 \le p < \infty$.

Proof Use smooth 'windows' $\psi_R \in S$, $0 \le \psi_R \le 1$, $\operatorname{supp}(\psi_R) = B(0,R)$, $\psi_R \equiv 1$ on $B(0,R-1)$, $R>1$. For any $\varepsilon > 0$, select $t>0$, such that $\|f-\varphi_t * f\|_p^p < \varepsilon/3$, and then $R>0$, such that $\|f\|^p_{L_p(\mathbb{R}^n\setminus B(0,R-1))}, \|\varphi_t * f\|^p_{L_p(\mathbb{R}^n\setminus B(0,R-1))} < \varepsilon/3$. Define $\phi := \psi_R(f*\varphi_t) \in S$.

$$\begin{aligned} \|f-\phi\|_p^p &= \int_{B(0,R-1)} |f - f*\varphi_t|^p + \int_{\mathbb{R}^n\setminus B(0,R-1)} |f-\psi(f*\varphi_t)|^p \\ &\le \int_{\mathbb{R}^n} |f-f*\varphi_t|^p + c\int_{\mathbb{R}^n\setminus B(0,R-1)} |f|^p + c\int_{\mathbb{R}^n\setminus B(0,R-1)} |f*\varphi_t|^p \\ &< \varepsilon \end{aligned}$$

□

Theorem For $\varphi,\psi \in S(\mathbb{R}^n)$

$$\langle \varphi,\psi\rangle_{L_2(\mathbb{R}^n)} = (2\pi)^{-n}\langle \hat{\varphi},\hat{\psi}\rangle_{L_2(\mathbb{R}^n)}.$$

Proof $(n=1)$. Let $g_\alpha(x) = \dfrac{1}{2\sqrt{\pi\alpha}} e^{-\frac{x^2}{4\alpha}}$. We already saw that $\hat{g}_\alpha(w) = e^{-\alpha w^2}$. We apply the special case of the inverse Fourier for Gaussians

$$\begin{aligned} \int \hat{g}_\alpha(w)\hat{\varphi}(w)\overline{\hat{\psi}(w)}dw &= \int \hat{g}_\alpha(w)\int \varphi(x)e^{-iwx}dx\int \overline{\psi(y)}e^{iwy}dydw \\ &= \int \varphi(x)\int \overline{\psi(y)}\left(\int \hat{g}_\alpha(w)e^{iw(y-x)}dw\right)dydx \\ &= 2\pi\int \varphi(x)\left(\int \overline{\psi(y)}g_\alpha(y-x)\,dy\right)dx \\ &= 2\pi\int \varphi(x)\left(\int \overline{\psi(y)}g_\alpha(x-y)\,dy\right)dx \end{aligned}$$

When we take limit $\alpha \to 0^+$

$$\int \hat{g}_\alpha(w)\hat{\varphi}(w)\overline{\hat{\psi}(w)}dw \quad = 2\pi\int \varphi(x)\left(\int \overline{\psi(y)}g_\alpha(x-y)dy\right)dx$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\int \hat{\varphi}(w)\overline{\hat{\psi}(w)}dw \qquad\qquad = 2\pi\int \varphi(x)\overline{\psi(x)}dx$$

□

Theorem [Inverse Fourier] For $\varphi \in S$

$$\varphi(x) = (\hat{\varphi})^{\vee}(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{\varphi}(w)e^{iwx}dw, \qquad \forall x \in \mathbb{R}^n.$$

Proof We use again the Gaussians. For $\alpha > 0$, by the previous theorem, for any $x \in \mathbb{R}^n$

$$\langle \varphi, g_\alpha(x-\cdot)\rangle = (2\pi)^{-n}\left\langle \hat{\varphi}, (g_\alpha(x-\cdot))^{\wedge}\right\rangle \Rightarrow$$

$$\int_{\mathbb{R}^n} \varphi(y)g_\alpha(x-y)dy = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{\varphi}(w)e^{iwx}\hat{g}_\alpha(w)dw \underset{\alpha\to 0}{\Rightarrow}$$

$$\varphi(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{\varphi}(w)e^{iwx}dw$$

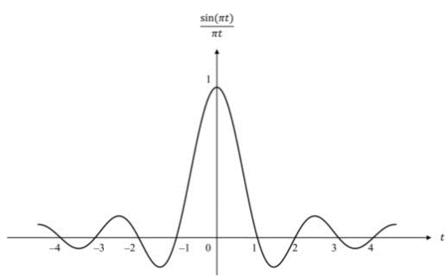
□

Theorem The Fourier transform is a homomorphism on the Schwartz class.

Definition Closure of $S \cap L_2$ in L_2 is L_2. So, we may extend the Fourier and inverse Fourier transform to L_2. Moreover, $\forall f, g \in L_2(\mathbb{R}^n)$ $\langle f, g\rangle = (2\pi)^{-n}\langle \hat{f}, \hat{g}\rangle$.

Example The sinc function. $\hat{f}(w) = 1_{[-\pi,\pi]}(w)$… what is f ?

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iwx}dw = \frac{1}{2\pi}\left.\frac{e^{iwx}}{ix}\right|_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{e^{i\pi x}-e^{-i\pi x}}{ix} = \frac{\sin \pi x}{\pi x}.$$



Fourier transform of distributions

Observe that for $\varphi, \psi \in S$

$$\int_{\mathbb{R}^n} \varphi(x)\hat{\psi}(x) = \int_{\mathbb{R}^n} \varphi(x)\left(\int_{\mathbb{R}^n} \psi(y)e^{-ixy}dy\right)dx$$

$$= \int_{\mathbb{R}^n} \psi(y)\left(\int_{\mathbb{R}^n} \varphi(x)e^{-ixy}dx\right)dy$$

$$= \int_{\mathbb{R}^n} \hat{\varphi}(y)\psi(y)dy$$

Again, we extend by duality

Def The Fourier transform of a distribution $f \in S'$ is defined by

$$\left\langle \hat{f},\varphi \right\rangle := \langle f,\hat{\varphi}\rangle, \qquad \forall \varphi \in S .$$

Example $f = \delta_0$

$$\left\langle \hat{\delta}_0,\varphi \right\rangle = \langle \delta_0,\hat{\varphi}\rangle = \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x)\,dx \Rightarrow (\delta_0)^{\wedge} \equiv 1 .$$

We can also show that $\left(\partial^\alpha \delta_0\right)^{\wedge}(w) = (iw)^\alpha$, $\alpha \in \mathbb{Z}_+^n$. Indeed,

$$\begin{aligned}
\left\langle \partial^\alpha \delta_0,\hat{\varphi} \right\rangle &:= (-1)^{|\alpha|}\left\langle \delta_0,\partial^\alpha \hat{\varphi} \right\rangle \\
&= (-1)^{|\alpha|}\,\partial^\alpha \hat{\varphi}(0) \\
&= (-1)^{|\alpha|}\,\partial_w^\alpha \left(\int_{\mathbb{R}^n} \varphi(x) e^{-iwx} dx \right)\Big|_{w=0} \\
&= \int_{\mathbb{R}^n} (ix)^\alpha\, \varphi(x) e^{-iwx} dx \Big|_{w=0} \\
&= \int_{\mathbb{R}^n} (ix)^\alpha\, \varphi(x)\, dx.
\end{aligned}$$

Few properties

(i) $f, g \in S'$, then $(f+g)^{\wedge} = \hat{f} + \hat{g}$. For any $\varphi \in S$

$$\left\langle (f+g)^{\wedge},\varphi \right\rangle = \langle f+g,\hat{\varphi}\rangle = \langle f,\hat{\varphi}\rangle + \langle g,\hat{\varphi}\rangle = \left\langle \hat{f},\varphi \right\rangle + \langle \hat{g},\varphi \rangle .$$

(ii) If $\varphi \in S$, $f \in S'$, then $(f * \varphi)^{\wedge} = \hat{f}\hat{\varphi}$.

<u>Definition of convolution:</u> Observe that for functions $\phi, \varphi, \psi \in S$

$$\begin{aligned}
\int_{\mathbb{R}^n} (\phi * \varphi)\psi &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \phi(y)\varphi(x-y)\,dy \right) \psi(x)\,dx \\
&= \int_{\mathbb{R}^n} \phi(y) \left(\int_{\mathbb{R}^n} \varphi(x-y)\psi(x)\,dx \right) dy \\
&= \int_{\mathbb{R}^n} \phi(y) \left(\int_{\mathbb{R}^n} \varphi(-(y-x))\psi(x)\,dx \right) dy \\
&= \int_{\mathbb{R}^n} \phi(y) \left(\varphi(-\cdot) * \psi \right)(y)\,dy
\end{aligned}$$

So, we define for $f \in S'$, $f * \varphi$ by

$$\langle f * \varphi, \psi \rangle := \langle f, \varphi(-\cdot) * \psi \rangle, \quad \psi \in S .$$

Alternatively

$$f * \varphi(x) := \langle f, \varphi(x-\cdot) \rangle, \qquad x \in \mathbb{R}^n .$$

<u>Definition of multiplication:</u> We define $f\varphi$ by

$$\langle f\varphi, \psi \rangle := \langle f, \varphi\psi \rangle, \quad \psi \in S .$$

Now,

$$\begin{aligned}
\left\langle (f * \varphi)^{\wedge}, \psi \right\rangle &= \langle f * \varphi, \hat{\psi} \rangle = \langle f, \varphi(-\cdot)\hat{\psi} \rangle \\
&= \left\langle \hat{f}, \left(\varphi(-\cdot)\hat{\psi}\right)^{\vee} \right\rangle \\
&= \left\langle \hat{f}, \hat{\varphi}\psi \right\rangle \\
&= \left\langle \hat{f}\hat{\varphi}, \psi \right\rangle
\end{aligned}$$

Function space characterization in the Fourier domain

Simple example: Parseval $f \in L_2(\mathbb{T}^n) \Leftrightarrow \sqrt{\sum_{k\in\mathbb{Z}^n} \left|\hat{f}(k)\right|^2} < \infty$, $\|f\|_{L_2(\mathbb{T}^n)} = \left\|\left\{\hat{f}(k)\right\}\right\|_{l_2(\mathbb{Z}^n)}$.

$f \in L_2(\mathbb{R}^n) \Leftrightarrow \hat{f} \in L_2(\mathbb{R}^n)$, $\|f\|_2 = (2\pi)^{-n}\left\|\hat{f}\right\|_2$.

Let's focus on the case of $\mathbb{R}^n$. We subdivide the frequencies to 'dyadic rings' $\Omega_j := \left\{w \in \mathbb{R}^n : 2^j < |w| \le 2^{j+1}\right\}$, $j \in \mathbb{Z}$. For $f \in L_2$, let f_j, be defined as the dyadic 'frequency slice' by $f_j = \left(\hat{f}\mathbf{1}_{\Omega_j}\right)^\vee$. We have that $f = \sum_j f_j$ and $\{f_j\}$ are orthogonal to each other because $\langle f_j, f_k\rangle = (2\pi)^{-n}\left\langle \hat{f}_j, \hat{f}_k\right\rangle = (2\pi)^{-n}\delta_{j,k}$. Therefore

$$\|f\|_2^2 = \left\langle \sum_j f_j, \sum_k f_k \right\rangle = \sum_j \langle f_j, f_j\rangle = \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |f_j|^2 \Rightarrow \|f\|_2 = \left\|\left(\sum_{j\in\mathbb{Z}} |f_j|^2\right)^{1/2}\right\|_2.$$

What about $p \neq 2$? Much more complicated, but we can mimic the above. Let $\varphi \in S$, such that $\operatorname{supp}(\hat{\varphi}) = B(0,2)$, $0 \le \hat{\varphi} \le 1$, $\hat{\varphi} \equiv 1$ on $B(0,1)$. Define $\psi \in S$, by $\hat{\psi}(w) := \hat{\varphi}(w) - \hat{\varphi}(2w)$ and $\hat{\psi}_j(w) := \hat{\psi}(2^{-j}w)$, $j \in \mathbb{Z}$. $\operatorname{supp}(\hat{\psi}_j) = \left\{w \in \mathbb{R}^n : 2^{j-1} \le |w| \le 2^{j+1}\right\}$

Lemma $\sum_j \hat{\psi}_j \equiv 1$.

Proof Let $w \in \mathbb{R}^n$. Select $J \in \mathbb{Z}$, such that $2^J < |w| \le 2^{J+1}$. This implies that $2^{-J-1}|w| \le 1$, $2^{-J+1}|w| \ge 2$.

$$\begin{aligned}
\sum_{j=-\infty}^{J-1} \hat{\psi}(2^{-j}w) &= \sum_{j=-\infty}^{J-1} \hat{\varphi}(2^{-j}w) - \hat{\varphi}(2^{-(j-1)}w) \\
&= \hat{\varphi}(2^{-J+1}w) - \hat{\varphi}(2^{-J+2}w) + \hat{\varphi}(2^{-J+2}w) - \hat{\varphi}(2^{-J+3}w) + \cdots \\
&= 0 - 0 + 0 - 0 + \cdots \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\sum_{j=J}^{\infty} \hat{\psi}(2^{-j}w) &= \sum_{j=J}^{\infty} \hat{\varphi}(2^{-j}w) - \hat{\varphi}(2^{-(j-1)}w) \\
&= \hat{\varphi}(2^{-J}w) - \hat{\varphi}(2^{-J+1}w) + \hat{\varphi}(2^{-J-1}w) - \hat{\varphi}(2^{-J}w) + \cdots \\
&= \underbrace{\hat{\varphi}(2^{-J-1}w)}_{=1} + \cdots \\
&= \hat{\varphi}(2^{-J-1}w) + \hat{\varphi}(2^{-J-2}w) - \hat{\varphi}(2^{-J-1}w) + \cdots \\
&= \underbrace{\hat{\varphi}(2^{-J-2}w)}_{=1} + \cdots \\
&= 1
\end{aligned}$$

□

Next, we define the operator $\Delta_j f := f * \psi_j$. Observe that since $\psi_j \in S$ $\{\Delta_j\}$ are well defined on S', so in particular on $L_p(\mathbb{R}^n)$, $0 < p \le \infty$. In the Fourier domain these operators serve as frequency 'cut-off' operators, similar to the simpler 'cut-off' indicators in L_2: $(\Delta_j f)^\wedge = \hat{f}\hat{\psi}_j$.

Theorem [Littlewood-Paley type] $f \in L_p\left(\mathbb{R}^n\right)$, $1 < p < \infty$ iff $\left(\sum_j \left|\Delta_j f\right|^2\right)^{1/2} \in L_p\left(\mathbb{R}^n\right)$.

$$\left\|\left(\sum_j \left|\Delta_j f\right|^2\right)^{1/2}\right\|_p = \left(\int_{\mathbb{R}^n} \left(\sum_j \left|\Delta_j f(x)\right|^2\right)^{p/2} dx\right)^{1/p}.$$

Theorem $f \in L_2\left(\mathbb{R}^n\right)$ is in $W_2^r\left(\mathbb{R}^n\right)$ iff

$$\left(\int_{\mathbb{R}^n} \left|\hat{f}(w)\right|^2 \left(1+|w|\right)^{2r} dw\right)^{1/2} < \infty ,$$

and

$$\|f\|_{r,2} \sim \left(\int_{\mathbb{R}^n} \left|\hat{f}(w)\right|^2 \left(1+|w|\right)^{2r} dw\right)^{1/2}$$

Remark This allows to defined **Fractional Sobolev spaces** for $s \in \mathbb{R}$, and $1 < p < \infty$ by checking for $f \in S'$ if

$$\left(\left(1+|\cdot|^2\right)^{s/2} \hat{f}\right)^{\vee} \in L_p .$$

Proof [One direction for now] First, let $f \in S$. We claim that for $1 \le k \le r$

$$\int_{\mathbb{R}^n} \left|\hat{f}(w)\right|^2 |w|^{2k} dw \sim |f|_{k,2}^2 = \left(\sum_{|\alpha|=k} \left\|\partial^\alpha f\right\|_2\right)^2$$

Let's start with $n=1$. In this case

$$\left(f^{(k)}\right)^{\wedge}(w) = (iw)^k \hat{f}(w) \Rightarrow \left|\left(f^{(k)}\right)^{\wedge}(w)\right| = |w|^k \left|\hat{f}(w)\right|.$$

So, by Parseval

$$|f|_{k,2}^2 = \left\|f^{(k)}\right\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}} \left|\left(f^{(k)}\right)^{\wedge}(w)\right|^2 dw = \frac{1}{2\pi}\int_{\mathbb{R}} \left|\hat{f}(w)\right|^2 |w|^{2k} dw .$$

For $n \ge 2$

$$|w|^{2k} = \left(\sum_{m=1}^n w_m^2\right)^k = \left(w_1^k\right)^2 + n\left(w_1^{k-1}w_2\right)^2 + \cdots + \left(w_n^k\right)^2 = \sum_{|\alpha|=k} a_\alpha \left(w^\alpha\right)^2 .$$

Repeated application, coordinate by coordinate, each step similar to the univariate case, gives

$$\left(\partial^\alpha f\right)^{\wedge}(w) = (iw)^\alpha \hat{f}(w) \Rightarrow \left|\left(\partial^\alpha f\right)^{\wedge}(w)\right| = \left|w^\alpha\right| \left|\hat{f}(w)\right| .$$

Example

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} f\right)^{\wedge}(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} f(x) e^{-iw_1x_1} e^{-iw_2x_2} dx_1 dx_2$$
$$= \int_{-\infty}^{\infty} e^{-iw_2x_2}\left(\int_{-\infty}^{\infty}\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2} f(x)\right) e^{-iw_1x_1} dx_1\right) dx_2$$
$$= iw_1 \int_{-\infty}^{\infty} e^{-iw_2x_2}\left(\int_{-\infty}^{\infty}\frac{\partial}{\partial x_2} f(x) e^{-iw_1x_1} dx_1\right) dx_2$$
$$= iw_1 \int_{-\infty}^{\infty} e^{-iw_1x_1}\left(\int_{-\infty}^{\infty}\frac{\partial}{\partial x_2} f(x) e^{-iw_2x_2} dx_2\right) dx_1$$
$$= -w_1 w_2 \int_{-\infty}^{\infty} e^{-iw_1x_1}\left(\int_{-\infty}^{\infty} f(x) e^{-iw_2x_2} dx_2\right) dx_1$$
$$= -w_1 w_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x) e^{-iw_1x_1} e^{-iw_2x_2} dx_1 dx_2$$
$$= -w_1 w_2 \hat{f}(w).$$

This gives

$$\left\|\partial^\alpha f\right\|_2^2 = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\left(\partial^\alpha f\right)^{\wedge}(w)\right|^2 dw = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\hat{f}(w)\right|^2 \left|w^\alpha\right|^2 dw .$$

Thus, we obtain

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\hat{f}(w)\right|^2 |w|^{2k} dw = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\hat{f}(w)\right|^2\left(\sum_{|\alpha|=k} a_\alpha \left|w^\alpha\right|^2\right) dw$$
$$\sim \sum_{|\alpha|=k}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\hat{f}(w)\right|^2\left|w^\alpha\right|^2 dw$$
$$\sim \sum_{|\alpha|=k}\left\|\partial^\alpha f\right\|_2^2$$
$$\underset{\|v\|_{l_1(K)}\sim\|v\|_{l_2(K)}}{\sim}\left(\sum_{|\alpha|=k}\left\|\partial^\alpha f\right\|_2\right)^2$$
$$\sim |f|_{k,2}^2 .$$

Therefore

$$\int_{\mathbb{R}^n}\left|\hat{f}(w)\right|^2 \left(1+|w|\right)^{2r} dw = \sum_{k=0}^{r} b_k \int_{\mathbb{R}^n}\left|\hat{f}(w)\right|^2 |w|^{2k} dw$$
$$\sim \sum_{k=0}^{r}\sum_{|\alpha|=k}\left\|\partial^\alpha f\right\|_2^2$$
$$\sim \left(\sum_{|\alpha|\le r}\left\|\partial^\alpha f\right\|_2\right)^2 = |f|_{r,2}^2 .$$

To complete the proof for $f \in W_2^r\left(\mathbb{R}^n\right)$, we apply density again. There is a sequence $\left\{\varphi_j\right\}_{j\ge1}$, $\varphi_j \in S$, such that $\left\|f-\varphi_j\right\|_{W_2^r} \underset{j\to\infty}{\to} 0$.

□

The Laplace operator, the Heat equation and Fourier transform

$\Omega = \mathbb{R}^n$, Laplace operator

$$L = -\Delta := -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} .$$

On $\mathbb{R}$ we have that $L\left(e^{iwx}\right) = \underbrace{w^2}_{eignevalue} \underbrace{e^{iwx}}_{eignefunction}$, $\forall w \in \mathbb{R}$. Spectral representation of the operator

$$Lf(x) = -\Delta f(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(f''\right)^{\wedge}(w) e^{iwx} dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} w^2 \hat{f}(w) e^{iwx} dw, \ \ \forall f \in W_2^2(\mathbb{R}) .$$

$$Lf(x) = -\Delta f(x) = -\Delta\left(\sum_k \hat{f}(k) e^{ikx}\right) = \sum_k k^2 \hat{f}(k) e^{ikx} , \qquad \forall f \in W_2^2(\mathbb{T}).$$

The Heat equation $u(x,t)$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, \\ u(x,0) = f(x). \end{cases}$$

The Gaussian (heat) Kernels satisfy the Heat equation

$$p_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} , \qquad \int_{\mathbb{R}^n} p_t(x)\, dx = 1, \qquad t > 0 .$$

Semi-group $p_t * p_s = p_{t+s}$, $t, s > 0$.

Theorem If f is continuous and bounded then

$$u(x,t) = p_t * f(x),$$

solves the Heat equation with initial conditions f .

Sketch Easy to see

$$\left(\frac{\partial}{\partial t} - \Delta\right) u(x,t) = \int_{\mathbb{R}^n} \underbrace{\left(\frac{\partial}{\partial t} - \Delta\right) p_t(x-y)}_{=0} f(y)\, dy = 0 .$$

$$u(x,t) = p_t * f(x) \underset{t \to 0}{\longrightarrow} f(x).$$

Spectral representation to solution of the Heat equation with boundary condition f

On $\mathbb{R}$ $$e^{-tL} f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-tw^2} \hat{f}(w) e^{iwx} dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(f * p_t\right)^{\wedge}(w) e^{iwx} dw = f * p_t(x) = u(x,t),$$

On $\mathbb{T}$ $$e^{-tL} f(x) = \sum_k e^{-tk^2} \hat{f}(k) e^{ikx} = \sum_k \left(p_t * f\right)^{\wedge}(k) e^{ikx} = p_t * f(x) = u(x,t).$$

What is the equivalent of the partial Fourier series on $\mathbb{R}^n$? Approximation from shift invariant spaces of the sinc (approximation theory course). Define $\varphi(u) = \mathbf{1}_{[-\pi,\pi]}(u)$. For any $h > 0$, we apply

$$\varphi\left(h\sqrt{L}\right) f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi\left(h|w|\right)\hat{f}(w) e^{iwx} dw$$
$$= \frac{1}{2\pi}\int_{-h^{-1}\pi}^{h^{-1}\pi} \hat{f}(w) e^{iwx} dw.$$

Maximal Functions (Stein, Chapter 1)

We work in a more general setting of a space X, with 'balls', $B(x,\delta)$, $x \in X$, $\delta > 0$, and measure μ. $\{B(x,\delta)\}$ are open sets in the topology of X. You can still think of the special case $X = \mathbb{R}^n$, $B(x,\delta) = \left\{y \in \mathbb{R}^n : \ |x-y| < \delta\right\}$, μ Lebesgue measure = volume.
We assume several properties:

(i) Monotonicity $B(x,\delta_1) \subseteq B(x,\delta_2)$, for $\delta_1 \le \delta_2$, $\forall x \in X$.

(ii) There exists $c_1 \ge 1$, such that $B(x,\delta) \cap B(y,\delta) \ne \emptyset \Rightarrow B(y,\delta) \subseteq B(x, c_1\delta)$.

(iii) There exists $c_2 \ge 1$, such that $|B(x,c_1\delta)| \le c_2 |B(x,\delta)|$ ("doubling condition").

(iv) $\bigcap_{\delta > 0} \bar{B}(x,\delta) = \{x\}$, $\forall x \in X$.

Example $X = \mathbb{R}^n$. We can choose, $c_1 = 3$ and $c_2 = c_1^n$, since
$$|B(x,c_1\delta)| = c(n)(c_1\delta)^n = c_1^n c(n)\delta^n = c_1^n |B(x,\delta)|.$$

Example for the construction of balls Let $\rho : X \times X \to \mathbb{R}_+$, be a quasi-distance satisfying with $\kappa \ge 1$

(i) $\rho(x,y) = 0 \Leftrightarrow x = y$

(ii) $\rho(x,y) = \rho(y,x)$

(iii) $\rho(x,y) \le \kappa\left(\rho(x,z) + \rho(y,z)\right)$

Then we can easily define balls by
$$B(x,\delta) := \left\{y \in X : \ \rho(x,y) < \delta\right\}.$$
These balls satisfy properties (i), (ii) above. To see (ii), assume $B(x,\delta) \cap B(y,\delta) \ne \emptyset$, then there exists $z \in B(x,\delta) \cap B(y,\delta)$. By the quasi-triangle inequality
$$\rho(x,y) \le \kappa\left(\rho(x,z) + \rho(y,z)\right) \le 2\kappa\delta.$$
If $w \in B(y,\delta)$, then
$$\rho(x,w) \le \kappa\left(\rho(x,y) + \rho(y,w)\right) \le \kappa(2\kappa\delta + \delta) = \underbrace{\kappa(2\kappa + 1)}_{c_1}\delta.$$
Observe that $c_1 = 3$, if $\kappa = 1 \Leftrightarrow \rho$ is a distance.

Definition $f \in L_1^{loc}(X)$, if for any compact $\Omega \subset X$, $\|f\|_{L_1(\Omega)} < \infty$.

Example Polynomials $P(x) = \sum_{|\alpha| < r} a_\alpha x^\alpha$, $P \notin L_1(\mathbb{R}^n)$, $P \in L_1^{loc}(\mathbb{R}^n)$

Definition For $f \in L_1^{loc}(X)$, we define the **centered maximal function**

$$Mf(x) := \sup_{\delta>0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y)| d\mu(y).$$

and the **uncentered maximal function**

$$\tilde{M}f(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| d\mu(y).$$

We claim that there exists a constant $c \geq 1$, such that

$$Mf(x) \leq \tilde{M}f(x) \leq cMf(x), \quad \forall x \in X.$$

Indeed, let $x \in B = B(y,\delta)$. This implies by property (ii) that $B(y,\delta) \subseteq B(x, c_1\delta)$.

$$\frac{1}{|B(y,\delta)|} \int_{B(y,\delta)} |f(y)| d\mu(y) \leq \underbrace{\frac{|B(y,c_1^2\delta)|}{|B(y,\delta)|}}_{\leq c_2^2} \underbrace{\frac{|B(x,c_1\delta)|}{|B(y,c_1^2\delta)|}}_{\leq 1 \text{ using (ii)}} \underbrace{\frac{1}{|B(x,c_1\delta)|} \int_{B(x,c_1\delta)} |f(y)| d\mu(y)}_{\leq Mf(x)}$$
$$\leq c_2^2 Mf(x).$$

Theorem [Maximal Function Theorem] Let $f : X \to \mathbb{C}$

(i) If $f \in L_1$, then for any $\alpha > 0$

$$|\{x \in X : Mf(x) > \alpha\}| \leq \frac{c}{\alpha} \|f\|_1 \Rightarrow \|Mf\|_{1,\infty} \leq c\|f\|_1$$

(ii) If $f \in L_p$, $1 < p \leq \infty$, then

$$\|Mf\|_p \leq A_p \|f\|_p,$$

with $A_p \sim \dfrac{1}{p-1}$, as $p \to 1$.

Example $X = \mathbb{R}$, $f = \mathbf{1}_{[-1,1]}$. Obviously $\|f\|_1 = 2$. Easy to see that for $x \in (-1,1)$, $Mf(x) = 1$. Why? Take $\delta > 0$ small enough such that $B(x,\delta) \subset (-1,1)$, then

$$\frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y)| dy = \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} dy = 1$$

However, for $x \notin [-1,1]$

$$Mf(x) = \sup_{\delta>0} \frac{1}{|(x-\delta, x+\delta)|} \int_{x-\delta}^{x+\delta} \mathbf{1}_{[-1,1]}(y) dy \underset{\delta=|x|+1}{=} \frac{1}{2(|x|+1)} \int_{[-1,1]} dy = \frac{1}{2(|x|+1)}.$$

And so $Mf \in L_p$, $1 < p \leq \infty$, but $Mf \notin L_1$. However, $Mf \in L_{1,\infty}$. To see that, first observe that $\|Mf\|_\infty \leq 1$. Thus, $\{x \in \mathbb{R} : Mf(x) > 1\} = \emptyset$. For any $0 < \alpha < 1/4$

$$\alpha |\{x \in \mathbb{R} : Mf(x) > \alpha\}| = 2\alpha + \alpha \left| \left\{ |x| \geq 1 : \frac{1}{2(|x|+1)} > \alpha \right\} \right|$$
$$\leq 2\alpha + 1 - 2\alpha = 1$$

For $1/4 < \alpha < 1$,

$$\alpha\left|\left\{x\in\mathbb{R}:\ Mf\left(x\right)>\alpha\right\}\right|=2\alpha+\alpha\left|\left\{|x|\ge 1:\ \frac{1}{2\left(|x|+1\right)}>\alpha\right\}\right|$$
$$\le 2\alpha+0\le 2$$

So, $\|f\|_{1,\infty}\le 2$.

To prove the theorem, we need Vitali's covering lemma

Lemma Under our assumptions on balls, let E be a finite union of balls. Then, one can select a pairwise disjoint subset $\left\{B_j\right\}_{j=1}^{J}$, such that

$$|E|\le c_2\sum_{j=1}^{J}\left|B_j\right|.$$

Proof Choose $B_1\left(x_1,\delta_1\right)$ as the ball of maximal radius. Next choose $B_2\left(x_2,\delta_2\right)$, such that $B_1\cap B_2=\varnothing$, with maximal radius $\delta_2\le\delta_1$. We continue the process until we can go no further. From our construction, the subset $\left\{B_j\right\}_{j=1}^{J}$ consists of pairwise disjoint balls. Any ball from the original set intersects with one of the balls $\left\{B_j\right\}_{j=1}^{J}$ otherwise it would have been added. By properties (i) and (ii), each ball $B_j\left(x_j,c_1\delta_j\right)$ contains all balls that intersect with $B_j\left(x_j,\delta_j\right)$, with radius $\le\delta_j$. Therefore, $E\subseteq\bigcup_{j=1}^{J}B_j\left(x_j,c_1\delta_j\right)$, and by property (iii)

$$|E|\le\left|\bigcup_{j=1}^{J}B_j\left(x_j,c_1\delta_j\right)\right|\le\sum_{j=1}^{J}\left|B_j\left(x_j,c_1\delta_j\right)\right|\le c_2\sum_{j=1}^{J}\left|B_j\right|.$$

□

Proof of maximal theorem It is sufficient to prove the theorem for the uncentered maximal function $\tilde{M}$. Denote by $E_\alpha:=\left\{x\in X:\ \tilde{M}f\left(x\right)>\alpha\right\}$. We <u>assume</u> that E_α is open! Let $E\subseteq E_\alpha$ be a compact subset. By definition, for each $x\in E$, there exists a ball B_x, such that $x\in B_x$ and

$$\alpha<\frac{1}{\left|B_x\right|}\int_{B_x}|f|\Rightarrow\left|B_x\right|<\frac{1}{\alpha}\int_{B_x}|f|.$$

Since E is compact, it can be covered by a finite collection of balls from $\left\{B_x\right\}_{x\in E}$. By the Vitali covering lemma, there exist a pairwise disjoint subset $\left\{B_j\right\}_{j=1}^{J}$, such that

$$|E|\le c_2\sum_{j=1}^{J}\left|B_j\right|.$$

This gives that

$$|E|\le c_2\sum_{j=1}^{J}\left|B_j\right|\le c_2\frac{1}{\alpha}\sum_{j=1}^{J}\int_{B_j}|f|\le c_2\frac{1}{\alpha}\int_X|f|.$$

We now <u>assume</u> that since E_α is open, it is a limit of a sequence of compact set $E\subseteq E_\alpha$ (this is true if $X=\mathbb{R}^n$, or is a σ-compact metric space). Therefore, we obtain (i)

$$\alpha\left|E_\alpha\right|=\alpha\left\{x\in X:\ \tilde{M}f\left(x\right)>\alpha\right\}\le c_2\int_X|f|,\quad\forall\alpha>0\quad\Rightarrow\|Mf\|_{1,\infty}\le c\|f\|_1.$$

We now prove (ii). For $p=\infty$, it is obvious that

$$\frac{1}{|B|}\int_B |f| \le \|f\|_\infty \Rightarrow Mf(x) \le \tilde{M}f(x) \le \|f\|_\infty .$$

Let $1 < p < \infty$. For $\alpha > 0$, let

$$f_1(x) := \begin{cases} f(x), & |f(x)| > \alpha/2, \\ 0, & \text{else.} \end{cases}$$

We have for $x \in B$

$$\begin{aligned} \frac{1}{|B|}\int_B |f| &= \frac{1}{|B|}\left(\int_{y\in B, |f(y)|>\alpha/2} |f(y)| d\mu(y) + \int_{y\in B, |f(y)|\le\alpha/2} |f(y)| d\mu(y)\right) \\ &\le \frac{1}{|B|}\int_B |f_1| + \frac{\alpha}{2} \\ &\le \tilde{M}f_1(x) + \frac{\alpha}{2}. \end{aligned}$$

This gives

$$\tilde{M}f(x) \le \tilde{M}f_1(x) + \frac{\alpha}{2} \Rightarrow \left\{x \in X : \tilde{M}f(x) > \alpha\right\} \subseteq \left\{x \in X : \tilde{M}f_1(x) > \alpha/2\right\}.$$

Next, we have

$$f \in L_p \Rightarrow \begin{cases} |\text{supp}(f_1)| < \infty \\ f_1 \in L_p,\ 1 < p \end{cases} \underset{\text{Lemma from lesson 1}}{\Rightarrow} f_1 \in L_1.$$

This means we may apply the first part of the theorem for f_1

$$\begin{aligned} \left|\left\{x \in X : \tilde{M}f(x) > \alpha\right\}\right| &\le \left|\left\{x \in X : \tilde{M}f_1(x) > \alpha/2\right\}\right| \\ &\le \frac{2c_2}{\alpha}\int_X |f_1| \\ &= \frac{2c_2}{\alpha}\int_{\{x:|f(x)|>\alpha/2\}} |f(x)| \end{aligned}$$

Finally,

$$\begin{aligned} \left\|\tilde{M}f\right\|_p^p &= p\int_0^\infty \left|\left\{x : \tilde{M}f(x) > \alpha\right\}\right| \alpha^{p-1} d\alpha \\ &\le 2c_2 p \int_0^\infty \alpha^{p-2}\left(\int_{\{x:|f(x)|>\alpha/2\}} |f|\right) d\alpha \\ &= 2c_2 p \int_X \left(\int_0^{2|f(x)|} \alpha^{p-2} d\alpha\right) |f(x)| d\mu(x) \\ &\underset{p>1}{=} \frac{c_2 p 2^p}{p-1}\int_X |f(x)|^{p-1} |f(x)| d\mu(x) \\ &= A_p \|f\|_p^p \end{aligned}$$

□

Hardy Spaces (Stein, Chapter 3)

Remark The Hardy spaces $H^p(\mathbb{R}^n)$ are equivalent to the $L_p(\mathbb{R}^n)$ spaces for $1 < p < \infty$. They are different from L_p, for $0 < p \le 1$. For many applications and from harmonic analysis perspective, they are the more appropriate choice for the range $0 < p \le 1$.

Fix $\varphi \in S$, $\int_{\mathbb{R}^n} \varphi = 1$. We define $\varphi_t(x) := t^{-n}\varphi(t^{-1}x)$. Easy to see that $\int_{\mathbb{R}^n} \varphi_t = 1$.

Definition For $f \in S'$, we define the **radial maximal function** $M_\varphi^\circ f(x) := \sup_{t>0} \left| f * \varphi_t(x) \right|$.

We are interested to investigate properties of such maximal functions. Why? Let's go back to the solution of the heat equation $u(x,t) = p_t * f(x)$, where f is the initial condition at $t = 0$.

Theorem Let φ is non-negative, radial $\left(\varphi(x) = \tilde{\varphi}(|x|), \tilde{\varphi} : \mathbb{R}_+ \to \mathbb{C}\right)$, and radially decreasing, with $\int_{\mathbb{R}^n} \varphi = 1$. Then, for any $f \in L_1^{loc}$

$$M_\varphi^0 f(x) \le Mf(x), \ \forall x \in \mathbb{R}^n .$$

Corollary If the initial condition f to the heat equation is in L_p, $1 < p \le \infty$, then the solution stays in L_p at all times. Indeed, using the above theorem together with the maximal theorem for the normalized Gaussian

$$\varphi(x) := \frac{1}{(4\pi)^{n/2}} e^{-|x|^2/4},$$

yields

$$\left\| u(\cdot,t') \right\|_p \le \left\| \sup_{t>0} \left| u(\cdot,t) \right| \right\|_p = \left\| M_\varphi^\circ f \right\|_p \le \left\| Mf \right\|_p \le C \left\| f \right\|_p .$$

Proof of the theorem

It is sufficient to prove $\left| f * \phi(x) \right| \le Mf(x)$, for any ϕ, non-negative, radial, radially decreasing with $\int_{\mathbb{R}^n} \phi = 1$, because we can then apply with $\phi = \varphi_t$, for any $t > 0$. Assume first $\phi(x) = \sum_{j=1}^N a_j \mathbf{1}_{B_j}(x)$, $a_j > 0$, $B_j = B(0, r_j)$.

We estimate

$$\begin{aligned}
\left| f * \mathbf{1}_{B_j}(x) \right| &\le \int_{\mathbb{R}^n} \left| f(y) \right| \mathbf{1}_{B_j}(x-y)\, dy \\
&= \underbrace{\int_{B(x,r_j)} \left| f(y) \right| dy}_{<\infty} \\
&= \underbrace{\left| B(x,r_j) \right|}_{=|B_j|} \frac{1}{\left| B(x,r_j) \right|} \int_{B(x,r_j)} \left| f(y) \right| dy \\
&\le \left| B_j \right| Mf(x).
\end{aligned}$$

Since $\int_{\mathbb{R}^n} \phi = \sum_{j=1}^N a_j \left| B_j \right| = 1$,

$$\begin{aligned}
\left| f * \phi(x) \right| &\le \sum_{j=1}^N a_j \left| f * \mathbf{1}_{B_j}(x) \right| \\
&\le Mf(x) \sum_{j=1}^N a_j \left| B_j \right| \\
&= Mf(x)
\end{aligned}$$

Now, any ϕ satisfying the required properties can be approximated by such radial 'step' functions.

□

Theorem For $1 < p \le \infty$, $f \in L_p \Leftrightarrow M_\varphi^\circ f \in L_p$.

Definition For $N \in \mathbb{N}$, we define $S_N := \{\varphi \in S: \ C_\varphi(\alpha, N) \le 1, \ |\alpha| \le N\}$. That is,

$$\left|\partial^\alpha \varphi(x)\right|\left(1+|x|\right)^N \le 1, \quad x \in \mathbb{R}^n, \ |\alpha| \le N.$$

Definition For $\varphi \in S$, $\int_{\mathbb{R}^n} \varphi = 1$, we define the **non-tangential maximal function**

$$M_\varphi f(x) = \sup_{t>0} \sup_{|y|<t} \left| f * \varphi_t(x-y) \right| = \sup_{t>0} \sup_{|x-z|<t} \left| f * \varphi_t(z) \right|.$$

It is easy to see that $M_\varphi^\circ f(x) \le M_\varphi f(x)$.

Definition We define the **grand radial maximal function**

$$M_N^\circ f(x) = \sup_{\varphi \in S_N} \sup_{t>0} \left| f * \varphi_t(x) \right|.$$

It is easy to see that $M_\varphi^\circ f(x) \le C(\varphi, N) M_N^\circ f(x)$, because

$$\frac{\varphi}{C(\varphi, N)} \in S_N, \qquad \text{where } C(\varphi, N) := \max_{|\alpha| \le N} C_\varphi(\alpha, N).$$

Definition Let $0 < p \le 1$. The **Hardy space** $H^p(\mathbb{R}^n)$ is defined as the set of tempered distributions $f \in S'$, such that with $N > n/p$, $\|f\|_{H^p} := \|M_N^\circ f\|_p < \infty$.

Example $f \in H^1(\mathbb{R})$

$$f(x) := \begin{cases} 1 & 0 \le x \le 1/2 \\ -1 & 1/2 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

The Hardy spaces are quasi-Banach spaces. Observe that $\|\cdot\|_{H^p}$, satisfies the quasi-triangle inequality

$$\begin{aligned} M_N^\circ(f+g)(x) &= \sup_{\varphi \in S_N} \sup_{t>0} \left| (f+g) * \varphi_t(x) \right| \\ &\le \sup_{\varphi \in S_N} \sup_{t>0} \left(\left| f * \varphi_t(x) \right| + \left| g * \varphi_t(x) \right| \right) \\ &\le \sup_{\varphi \in S_N} \sup_{t>0} \left| f * \varphi_t(x) \right| + \sup_{\tilde{\varphi} \in S_N} \sup_{\tilde{t}>0} \left| g * \tilde{\varphi}_{\tilde{t}}(x) \right| \\ &\le M_N^\circ f(x) + M_N^\circ g(x). \end{aligned}$$

So,

$$\begin{aligned} \left\| M_N^\circ(f+g) \right\|_p^p &\le \left\| M_N^\circ f + M_N^\circ g \right\|_p^p \\ &\le \left\| M_N^\circ f \right\|_p^p + \left\| M_N^\circ g \right\|_p^p. \end{aligned}$$

Theorem Let $0 < p \le 1$. Then for any $f \in S'$, and $\varphi \in S$, $\int_{\mathbb{R}^n} \varphi = 1$, and sufficiently large $N > n/p$

$$\left\| M_N^\circ f \right\|_p \sim \left\| M_\varphi^\circ f \right\|_p \sim \left\| M_\varphi f \right\|_p.$$

We shall prove parts of the theorem, using a series of results.

Lemma Let $\phi \in S$, then $\hat{\phi} \in S$, with $C_{\hat{\phi}}(\alpha, N) \le \tilde{C} \sum_{|\beta| \le |\alpha| + N} C_\phi(\beta, n+1)$. Also $\phi^\vee \in S$.

Sketch of proof Let's demonstrate with the univariate case. Let's estimate the decay of $\hat{\phi}$. For $w \in \mathbb{R}$, $|w| < 1$

$$\left|\hat{\phi}(w)\right| \le \int_{-\infty}^{\infty} |\phi(x)| dx = \int_{-\infty}^{\infty} |\phi(x)| (1+|x|)^2 (1+|x|)^{-2} dx \le CC_\phi(0,2).$$

For $w \in \mathbb{R}$, $|w| \ge 1$

$$\hat{\phi}(w) = \int_{-\infty}^{\infty} \phi(x) e^{-iwx} dx = \underbrace{\phi(x) \frac{e^{-iwx}}{-iw}\Bigg|_{-\infty}^{\infty}}_{=0} + \frac{1}{iw} \int_{-\infty}^{\infty} \phi'(x) e^{-iwx} dx = \frac{1}{iw} (\phi')^{\wedge}(w).$$

After r times,

$$\left|\hat{\phi}(w)\right| \le |w|^{-r} \int_{-\infty}^{\infty} \left|\phi^{(r)}(x)\right| dx = |w|^{-r} \int_{-\infty}^{\infty} \left|\phi^{(r)}(x)\right| (1+|x|)^2 (1+|x|)^{-2} dx \le CC_\phi(r,2)|w|^{-r}.$$

And so,

$$C_{\hat{\phi}}(0,r) \le C\left(C_\phi(0,2) + C_\phi(r,2)\right).$$

Now, the derivative of $\hat{\phi}$

$$\frac{d}{dw}\hat{\phi}(w) = \frac{d}{dw}\left(\int_{-\infty}^{\infty} \phi(x) e^{-iwx} dx\right) = \int_{-\infty}^{\infty} \left(-ix\phi(x)\right) e^{-iwx} dx.$$

For $w \in \mathbb{R}$, $|w| < 1$

$$\left|\frac{d}{dw}\hat{\phi}(w)\right| \le \int_{-\infty}^{\infty} |x\phi(x)| dx = \int_{-\infty}^{\infty} |\phi(x)| (1+|x|)^3 \frac{|x|}{1+|x|} (1+|x|)^{-2} dx$$

$$\le C_\phi(0,3) \int_{-\infty}^{\infty} (1+|x|)^{-2} dx = CC_\phi(0,3).$$

For $w \in \mathbb{R}$, $|w| \ge 1$

$$\begin{aligned}
\frac{d}{dw}\hat{\phi}(w) &= \int_{-\infty}^{\infty} \left(-ix\phi(x)\right) e^{-iwx} dx \\
&= \underbrace{\left(-ix\phi(x)\right) \frac{e^{-iwx}}{-iw}\Bigg|_{-\infty}^{\infty}}_{=0} + \frac{1}{iw} \int_{-\infty}^{\infty} \left(-ix\phi(x)\right)' e^{-iwx} dx \\
&= \frac{1}{w}\left(\int_{-\infty}^{\infty} \phi(x) e^{-iwx} dx + \int_{-\infty}^{\infty} x\phi'(x) e^{-iwx} dx\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\frac{d}{dw}\hat{\phi}(w)\right| &\le \frac{1}{|w|}\left(\int_{-\infty}^{\infty} |\phi(x)| dx + \int_{-\infty}^{\infty} |x\phi'(x)| dx\right) \\
&= \frac{1}{|w|}\left(\int_{-\infty}^{\infty} |\phi(x)| (1+|x|)^2 (1+|x|)^{-2} dx + \int_{-\infty}^{\infty} |\phi'(x)| (1+|x|)^3 \frac{|x|}{1+|x|} (1+|x|)^{-2} dx\right) \\
&\le \frac{C}{|w|}\left(C_\phi(0,2) + C_\phi(1,3)\right).
\end{aligned}$$

$\square$

The next theorem allows to pass from one Schwartz function to another.

Theorem Let $\varphi, \phi \in S$, with $\int_{\mathbb{R}^n} \varphi = 1$. Then there exists a sequence $\{\eta_k\}_{k=0}^{\infty}$, $\eta_k \in S$, such that

$$\phi = \sum_{k=0}^{\infty} \eta_k * \varphi_{2^{-k}} ,$$

where η_k are 'rapidly decreasing' with k, in the following sense: for any $M > 0$, $\alpha \in \mathbb{Z}_+^n$, and $N > 0$,

$$C_{\eta_k}(\alpha, N) \le C(M, \alpha, N, \varphi, \phi) 2^{-Mk} .$$

Proof Recall the frequency-side smooth windows. We defined $\psi \in S$, and then $\hat{\psi}_k(w) := \hat{\psi}(2^{-k} w)$, $k \in \mathbb{Z}$, $\text{supp}(\hat{\psi}_k) = \{w \in \mathbb{R}^n : 2^{k-1} \le |w| \le 2^{k+1}\}$, with $\sum_{k=-\infty}^{\infty} \hat{\psi}_k \equiv 1$. One can modify by selecting $\hat{\psi}_0$ $\text{supp}(\hat{\psi}_0) = \{w \in \mathbb{R}^n : 0 \le |w| \le 2\}$, and then $\sum_{k=0}^{\infty} \hat{\psi}_k \equiv 1$. Therefore, we have

$$\hat{\phi} = \sum_{k=0}^{\infty} \hat{\psi}_k \hat{\phi} .$$

Under the assumption that $\int \varphi = \hat{\varphi}(0) = 1$, and from the continuity of $\hat{\varphi}$, we may assume for a moment that $|\hat{\varphi}(w)| \ge 1/2$, for $|w| \le 2$. This allows to write (under the assumption 0/0=0)

$$\hat{\phi}(w) = \sum_{k=0}^{\infty} \underbrace{\frac{\hat{\psi}_k(w)}{\hat{\varphi}(2^{-k} w)} \hat{\phi}(w)}_{\hat{\eta}_k} \hat{\varphi}(2^{-k} w) = \sum_{k=0}^{\infty} \hat{\eta}_k(w) \hat{\varphi}(2^{-k} w).$$

For each $k \ge 1$,

$$w \in \text{supp}(\hat{\psi}_k) \Rightarrow 2^{k-1} \le |w| \le 2^{k+1} \Rightarrow 2^{-1} \le 2^{-k}|w| \le 2 \Rightarrow |\hat{\varphi}(2^{-k} w)| \ge 1/2 .$$

So, one can see that $\hat{\eta}_k \in S \Rightarrow \eta_k \in S$. Observe that

$$(\varphi_{2^{-k}})^{\wedge}(w) = 2^{kn} \int_{\mathbb{R}^n} \varphi(2^k x) e^{-iwx} dx \underset{y=2^k x}{=} \int_{\mathbb{R}^n} \varphi(y) e^{-iw2^{-k} y} dy = \hat{\varphi}(2^{-k} w).$$

This gives

$$\phi = \sum_{k=0}^{\infty} \eta_k * \varphi_{2^{-k}} .$$

We now show that $\{\eta_k\}$ are 'rapidly decreasing' with k. It is sufficient to show this for $\{\hat{\eta}_k\}$.
Let's look at the frequency side

$$\text{supp}(\hat{\eta}_k) = \text{supp}\left(\frac{\hat{\psi}_k}{\hat{\varphi}(2^{-k} \cdot)} \hat{\phi} \right) = \{w \in \mathbb{R}^n : 2^{k-1} \le |w| \le 2^{k+1}\} .$$

By the previous lemma $\hat{\phi} \in S$, with Schwartz constants that depend on the constants of ϕ. So, for any $L > 0$, $w \in \text{supp}(\hat{\eta}_k)$

$$\begin{aligned} |\hat{\eta}_k(w)| &\le 2|\hat{\phi}(w)| \\ &= 2|\hat{\phi}(w)| (1+|w|)^{M+L} (1+|w|)^{-(M+L)} \\ &\le 2C_{\hat{\phi}}(0, M+L)(1+|w|)^{-L} (1+2^{k-1})^{-M} \\ &\le C(\phi, M+L)(1+|w|)^{-L} 2^{-kM}. \end{aligned}$$

So

$$C_{\hat{\eta}_k}(0, L) \le C(\phi, M+L) 2^{-kM}.$$

In similar manner, for the derivatives.

We now deal with the assumption that $|\hat{\varphi}(w)| \geq 1/2$, for $|w| \leq 2$. By continuity of $\hat{\varphi}$, there exists $k_0 \geq 0$, such that $|\hat{\varphi}(2^{-k_0} w)| \geq 1/2$, for $|w| \leq 2$. Apply the proof to $\Phi := (\hat{\phi}(2^{-k_0} \cdot))^{\vee} = 2^{k_0 n}\phi(2^{k_0} \cdot)$

$$2^{k_0 n}\phi(2^{k_0} x) = \Phi(x) = \sum_{k=0}^{\infty} \tilde{\eta}_k * \varphi_{2^{-k}}(x) \Rightarrow$$

$$\phi(x) = \sum_{k=0}^{\infty} \underbrace{2^{-k_0 n}\tilde{\eta}_k}_{\eta_{k+k_0}} * \varphi_{2^{-k}}(2^{k_0} x) \Rightarrow$$

$$\phi(x) = \sum_{k=k_0}^{\infty} \eta_k * \varphi_{2^{-k}}(x) \Rightarrow$$

$$\phi(x) = \sum_{k=0}^{\infty} \eta_k * \varphi_{2^{-k}}(x), \quad \eta_k := 0,\ 0 \leq k < k_0.$$

□

Definition We define for $\varphi \in S$ and $N \in \mathbb{N}$

$$T_{\varphi}^{N} f(x) = \sup_{t>0} \sup_{y \in \mathbb{R}^n} |f * \varphi_t(x-y)| \left(1 + \frac{|y|}{t}\right)^{-N}.$$

Observe that

$$\begin{aligned} M_{\varphi} f(x) &= \sup_{t>0} \sup_{|y|<t} |f * \varphi_t(x-y)| \\ &= \sup_{t>0} \sup_{|y|<t} |f * \varphi_t(x-y)| \left(1 + \frac{|y|}{t}\right)^{-N} \left(1 + \frac{|y|}{t}\right)^{N} \\ &\leq 2^N \sup_{t>0} \sup_{y \in \mathbb{R}^n} |f * \varphi_t(x-y)| \left(1 + \frac{|y|}{t}\right)^{-N} \\ &= 2^N T_{\varphi}^{N} f(x) \end{aligned}$$

The next lemma is the inverse of the above

Lemma If $M_{\varphi} f \in L_p(\mathbb{R}^n)$, and $N > n/p$, then

$$\|T_{\varphi}^{N} f\|_p \leq c(N, p, \varphi) \|M_{\varphi} f\|_p.$$

Theorem Let $\varphi \in S$, with $\int_{\mathbb{R}^n} \varphi = 1$. Then, for sufficiently large $N > n/p$

$$\|M_N^{\circ} f\|_p \leq c \|M_{\varphi} f\|_p.$$

Proof Let $\phi \in S_N$. Then, by a previous theorem, there exists a sequence $\{\eta_k\}$ with 'fast decreasing properties with k' such that

$$\phi = \sum_{k=0}^{\infty} \eta_k * \varphi_{2^{-k}}.$$

This gives

$$M_{\phi}^{\circ} f(x) = \sup_{t>0} |f * \phi_t(x)| \leq \sup_{t>0} \sum_{k=0}^{\infty} \left| f * (\eta_k * \varphi_{2^{-k}})_t(x) \right|.$$

Observe that (assignment)

$$\left(\eta_k * \varphi_{2^{-k}}\right)_t(z) = \left(\eta_k\right)_t * \varphi_{2^{-k}t}(z).$$

This allows to estimate

$$\begin{aligned}
M^{\circ}_{\phi} f(x) &\le \sup_{t>0} \sum_{k=0}^{\infty} \left| f * (\eta_k)_t * \varphi_{2^{-k}t}(x) \right| \\
&\le \sup_{t>0} \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \left| f * \varphi_{2^{-k}t}(x-y) \right| \left| (\eta_k)_t(y) \right| dy \\
&= \sup_{t>0} \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \underbrace{\left| f * \varphi_{2^{-k}t}(x-y) \right| \left(1 + \frac{|y|}{2^{-k}t}\right)^{-N}}_{\le T^N_{\varphi}(x)} \left(1 + \frac{|y|}{2^{-k}t}\right)^{N} \left| (\eta_k)_t(y) \right| dy \\
&\le T^N_{\varphi}(x) \sup_{t>0} \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \left(1 + \frac{|y|}{2^{-k}t}\right)^{N} \left| (\eta_k)_t(y) \right| dy
\end{aligned}$$

For any $t > 0$

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(1 + \frac{|y|}{2^{-k}t}\right)^{N} \left| (\eta_k)_t(y) \right| dy &= t^{-n} \int_{\mathbb{R}^n} \left(1 + \frac{|y|}{2^{-k}t}\right)^{N} \left| \eta_k\left(t^{-1}y\right) \right| dy \\
&\underset{z=t^{-1}y}{=} \int_{\mathbb{R}^n} \left(1 + 2^k |z|\right)^N \left| \eta_k(z) \right| dz \\
&\le C 2^{kN} C_{\eta_k}(0, N+n+1) \\
&\underset{M=N+1}{\le} C 2^{kN} 2^{-k(N+1)} \\
&\le C 2^{-k}
\end{aligned}$$

We get

$$M^{\circ}_{\phi} f(x) \le C T^N_{\varphi}(x).$$

Finally, using the previous lemma

$$\left\| M^{\circ}_{N} f \right\|_p = \sup_{\phi \in S_N} \left\| M^{\circ}_{\phi} f \right\|_p \le C \left\| T^N_{\varphi} f \right\|_p \le C \left\| M_{\varphi} f \right\|_p,$$

where for sufficiently large N, the constant does not depend on the choice of ϕ (sufficiently high order Schwarz constants are normalized).

□

Theorem Let $\varphi \in S$, with $\int_{\mathbb{R}^n} \varphi = 1$. Then,

$$\left\| M_{\varphi} f \right\|_p \le c \left\| M^{\circ}_{\varphi} f \right\|_p, \quad f \in S'.$$

Proof We will prove the theorem under the <u>assumption</u> that $\left\| M_{\varphi} f \right\|_p < \infty$. There are quite a few technicalities that are required to remove this assumption (see e.g. Stein). For $\lambda > 0$, let

$$\Omega_{\lambda} := \left\{ x \in \mathbb{R}^n : M^{\circ}_{N} f(x) \le \lambda M_{\varphi} f(x) \right\}.$$

Take $\lambda^p \ge 2c^p$, where $c > 0$ is from the previous theorem $\left(\left\| M^{\circ}_{N} f \right\|_p \le c \left\| M_{\varphi} f \right\|_p \right)$. Then,

$$\int_{\Omega_\lambda^c} \left(M_\varphi f\right)^p \le \lambda^{-p} \int_{\Omega_\lambda^c} \left(M_N^\circ f\right)^p$$
$$\le \lambda^{-p} \int_{\mathbb{R}^n} \left(M_N^\circ f\right)^p$$
$$\le c^p \lambda^{-p} \int_{\mathbb{R}^n} \left(M_\varphi f\right)^p \le \frac{1}{2} \int_{\mathbb{R}^n} \left(M_\varphi f\right)^p .$$

This gives (under the assumption!)

$$\int_{\mathbb{R}^n} \left(M_\varphi f\right)^p = \int_{\Omega_\lambda} \left(M_\varphi f\right)^p + \int_{\Omega_\lambda^c} \left(M_\varphi f\right)^p$$
$$\le \int_{\Omega_\lambda} \left(M_\varphi f\right)^p + \frac{1}{2} \int_{\mathbb{R}^n} \left(M_\varphi f\right)^p \Rightarrow$$
$$\int_{\mathbb{R}^n} \left(M_\varphi f\right)^p \le 2 \int_{\Omega_\lambda} \left(M_\varphi f\right)^p$$

Assume that for any $q > 0$ and $x \in \Omega_\lambda$

$$M_\varphi f(x) \le c \left[M \left(M_\varphi^\circ f\right)^q (x) \right]^{1/q} . \qquad (*)$$

Then, we can take $0 < q < p$ and apply the maximal function theorem for $r := p/q > 1$

$$\int_{\mathbb{R}^n} \left(M_\varphi f(x)\right)^p dx \le 2 \int_{\Omega_\lambda} \left(M_\varphi f(x)\right)^p dx$$
$$\le C \int_{\Omega_\lambda} \left[M \left(M_\varphi^\circ f\right)^q (x) \right]^{p/q} dx$$
$$\le C \int_{\mathbb{R}^n} \left[M \left(M_\varphi^\circ f\right)^q (x) \right]^{p/q} dx$$
$$\le C \int_{\mathbb{R}^n} \left[\left(M_\varphi^\circ f\right)^q (x) \right]^{p/q} dx$$
$$= C \int_{\mathbb{R}^n} \left(M_\varphi^\circ f(x)\right)^p dx.$$

It remains to prove (*). Let $F^\circ(y,t) := f * \varphi_t(y)$. For any $x \in \mathbb{R}^n$, there exist $y \in \mathbb{R}^n$, $t > 0$, $|x-y| < t$, such that

$$\left|F^\circ(y,t)\right| \ge \sup_{t>0} \sup_{|x-y|<t} \left| f * \varphi_t(y) \right| / 2 = M_\varphi(x)/2 .$$

For sufficiently small $r > 0$ (to be chosen later), and $x' \in B(y, rt)$,

$$\left|F^\circ(x',t) - F^\circ(y,t)\right| \le rt \sup_{z \in B(y,rt)} \left|\nabla F^\circ(z,t)\right| .$$

Observe that

$$\frac{\partial}{\partial z_i} F^\circ(z,t) = f * \frac{\partial}{\partial z_i} \varphi_t(z) = t^{-1} f * \left(\frac{\partial \varphi}{\partial z_i}\right)_t (z), \quad 1 \le i \le n .$$

For the set of functions

$$\mathcal{F} := \left\{ \frac{\partial \varphi}{\partial z_i}(\cdot + h) : \ 1 \le i \le n, \ |h| \le 1 + r \right\},$$

the Schwartz constants of order $\le N$ are uniformly bounded by a constant depending on φ and r. Since for $z \in B(y, rt)$, $t^{-1}|x-z| \le t^{-1}\left(|x-y| + |y-z|\right) \le 1 + r$

$$\left| f*\left(\frac{\partial\varphi}{\partial z_i}\right)_t(z)\right| = \left| f*\left(\frac{\partial\varphi}{\partial z_i}\left(\cdot + t^{-1}(z-x)\right)\right)_t(x)\right|$$

$$\le \sup_{\psi\in\mathcal{F}}\sup_{t>0}\left| f*\psi_t(x)\right|$$

$$\le cM_N^{\circ} f(x).$$

Thus, for $x\in\Omega_\lambda$

$$\left|F^{\circ}(x',t)-F^{\circ}(y,t)\right| \le crM_N^{\circ} f(x)$$

$$\le c\lambda r M_\varphi f(x)$$

Now choose r sufficiently small such that $c\lambda r \le 1/4$. Therefore

$$\left.\begin{array}{r}\left|F^{\circ}(y,t)\right| \ge M_\varphi f(x)/2 \\ \left|F^{\circ}(x',t)-F^{\circ}(y,t)\right| \le M_\varphi f(x)/4\end{array}\right\} \Rightarrow \left|F^{\circ}(x',t)\right| \ge \frac{M_\varphi f(x)}{4}, \quad x'\in B(y,rt).$$

Therefore since $B(y,rt)\subset B(x,(1+r)t)$

$$\left(M_\varphi f(x)\right)^q \le \frac{4^q}{\left|B(y,rt)\right|}\int_{B(y,rt)}\left|F^{\circ}(x',t)\right|^q dx'$$

$$\le C\frac{\left|B(x,t(1+r))\right|}{\left|B(y,rt)\right|}\frac{1}{\left|B(x,t(1+r))\right|}\int_{B(x,t(1+r))}\left|F^{\circ}(x',t)\right|^q dx'$$

$$\le C\frac{(1+r)^n}{r^n}M\left(M_\varphi^{\circ} f\right)^q(x)$$

$$\le CM\left(M_\varphi^{\circ} f\right)^q(x).$$

□

Atomic Hardy Spaces

Definition A function $a:\mathbb{R}^n\to\mathbb{C}$, is an atom for $0<p\le 1$, if

(i) $\operatorname{supp}(a)\subseteq B$, for some ball B,

(ii) $\|a\|_\infty \le |B|^{-1/p}$,

(iii) $\int x^\alpha a(x)\,dx = 0$, $\quad \forall\alpha\in\mathbb{Z}_+^n$, $|\alpha|\le n\left(p^{-1}-1\right)$. (vanishing moments property)

Notice that

$$\|a\|_p = \left(\int_B |a|^p\right)^{1/p} \le \|a\|_\infty |B|^{1/p} \le |B|^{-1/p}|B|^{1/p} = 1.$$

Theorem For any atom a, we have that $\|a\|_{H^p}\le c$.

Back to the example $\mathbf{1}_{[0,1]}(\cdot)\notin H^1(\mathbb{R})$ … however $f\in H^1(\mathbb{R})$

$$f(x) := \begin{cases} 1 & 0 \le x \le 1/2 \\ -1 & 1/2 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

Background on multivariate Taylor polynomials

The multivariate Taylor polynomial is given by

$$T_{r-1,\bar{x}}\, g(y) := \sum_{|\alpha|<r} \frac{\partial^\alpha g(\bar{x})}{\alpha!} (y-\bar{x})^\alpha \in \Pi_{r-1} \ ,$$

The estimate of Taylor remainder

$$\left| g(y) - T_{r-1,\bar{x}}\, g(y) \right| = \left| R_{r,\bar{x}}\, g(y) \right| \le c \left| y - \bar{x} \right|^r \max_{z \in B(\bar{x}, |y-\bar{x}|)} \max_{|\alpha|=r} \left| \partial^\alpha g(z) \right| \ .$$

Proof Assume a is an atom supported on $B = B(\bar{x}, r)$. Let $\varphi \in S$, $\int \varphi \ne 0$, with $\operatorname{supp}(\varphi) = B(0,1)$. Since $\left\| M_N^\circ a \right\|_p \le c \left\| M_\varphi^\circ a \right\|_p$, it is sufficient to bound $\left\| M_\varphi^\circ a \right\|_p$. The first estimate is that

$$\left| a * \varphi_t(x) \right| \le \int_{\mathbb{R}^n} \left| a(y) \right| \left| \varphi_t(x-y) \right| dy \le \left\| \varphi \right\|_1 \left| B \right|^{-1/p}, \quad \forall x \in \mathbb{R}^n .$$

We use it 'near' the ball on $B_2 := B(\bar{x}, 2r)$

$$\int_{B_2} \left(M_\varphi^\circ a \right)^p \le c \left| B \right|^{-1} \left| B_2 \right| = c \frac{(2r)^n}{r^n} = c' \ .$$

We now estimate 'away' from the ball. Let $x \notin B_2$. Observe that for $t > 0$, the support of φ_t is $B(0,t)$. If $\left| x - \bar{x} \right| > r + t$, then $\operatorname{supp}\left(\varphi(x - \cdot) \right) \cap B = \emptyset$, and

$$a * \varphi_t(x) = \int_{\mathbb{R}^n} a(y) \varphi_t(x-y) dy = \int_B a(y) \varphi_t(x-y) dy = 0 \ .$$

Otherwise, we assume $\left| x - \bar{x} \right| \le r + t$, and use the vanishing moments of a

$$a * \varphi_t(x) = \int_{\mathbb{R}^n} a(y) \varphi_t(x-y) dy = \int_B a(y) \left(\varphi_t(x-y) - q_{x,t}(y) \right) dy \ ,$$

where

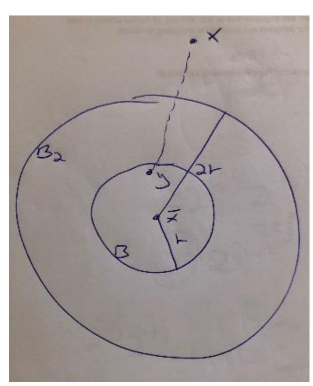
$$q_{x,t}(y) := T_{d,\bar{x}} \left(\varphi_t(x - \cdot) \right)(y), \quad d := \left\lfloor n \left(p^{-1} - 1 \right) \right\rfloor .$$

By the estimate of the Taylor remainder

$$\left| \varphi_t(x-y) - q_{x,t}(y) \right| \le c \left| y - \bar{x} \right|^{d+1} \max_{|\alpha|=d+1} \left\| \partial^\alpha \varphi_t \right\|_\infty = c \frac{\left| y - \bar{x} \right|^{d+1}}{t^{n+d+1}} \ .$$

So,

$$\left| x - \bar{x} \right| \ge 2r \Rightarrow \quad \left| x - \bar{x} \right| \le t + r \le t + \left| x - \bar{x} \right| / 2 \ \Rightarrow \frac{\left| x - \bar{x} \right|}{2} \le t \ .$$



We get

$$\left|\varphi_t\left(x-y\right)-q_{x,t}\left(y\right)\right| \le C\frac{\left|y-\overline{x}\right|^{d+1}}{\left|x-\overline{x}\right|^{n+d+1}},$$

and

$$\begin{aligned}M_{\varphi}^{\circ}a\left(x\right) &= \sup_{t>0}\left|a*\varphi_t\left(x\right)\right| \\ &\le \sup_{t>0}\int_B\left|a\left(y\right)\right|\left|\varphi_t\left(x-y\right)-q_{x,t}\left(y\right)\right|dy \\ &\le C\left|B\right|^{-1/p}\int_B\frac{\left|y-\overline{x}\right|^{d+1}}{\left|x-\overline{x}\right|^{n+d+1}}dy \\ &\le C\left|B\right|^{-1/p}\left(\frac{r}{\left|x-\overline{x}\right|}\right)^{n+d+1}.\end{aligned}$$

Note

$$\begin{aligned}&d := \left\lfloor n\left(p^{-1}-1\right)\right\rfloor > n\left(p^{-1}-1\right)-1 \Rightarrow \\ &d+1 > n\left(p^{-1}-1\right) \Rightarrow \\ &p\left(n+d+1\right) > p\left(n+n\left(p^{-1}-1\right)\right) > n\end{aligned}$$

We obtain

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_2}\left|M_{\varphi}^{\circ}a\left(x\right)\right|^p dx &\le C\left|B\right|^{-1}r^{p(n+d+1)}\int_{\mathbb{R}^n\setminus B(\overline{x},2r)}\left|x-\overline{x}\right|^{-p(n+d+1)}dx \\ &= C\left|B\right|^{-1}r^{p(n+d+1)}\int_{\mathbb{R}^n\setminus B(0,2r)}\left|x\right|^{-p(n+d+1)}dx \\ &\le C\left|B\right|^{-1}r^{p(n+d+1)}\left(2r\right)^{n-p(n+d+1)} \\ &\le Cr^{-n}r^n = C.\end{aligned}$$

□

Definition The atomic Hardy space $H_a^p\left(\mathbb{R}^n\right)$, is the set of all distributions $f=\sum_{k=1}^{\infty}\lambda_k a_k$, such that $\{a_k\}$ are atoms and $\sum_{k=1}^{\infty}\left|\lambda_k\right|^p<\infty$. We set

$$\left\|f\right\|_{H_a^p} := \inf\left\{\left(\sum_{k=1}^{\infty}\left|\lambda_k\right|^p\right)^{1/p} : \ f=\sum_{k=1}^{\infty}\lambda_k a_k\right\}.$$

Theorem $H^p\left(\mathbb{R}^n\right)\sim H_a^p\left(\mathbb{R}^n\right)$

Proof of easy direction Let $f\in H_a^p\left(\mathbb{R}^n\right)$, then $f=\sum_{k=1}^{\infty}\lambda_k a_k$, with $\sum_{k=1}^{\infty}\left|\lambda_k\right|^p<2\left\|f\right\|_{H_a^p}^p$. As we have seen, the maximal functions are sublinear. Therefore, by the previous theorem

$$\left\|M_N^{\circ}f\right\|_p^p \le \sum_{k=1}^{\infty}\left|\lambda_k\right|^p\left\|M_N^{\circ}a_k\right\|_p^p \le c\sum_{k=1}^{\infty}\left|\lambda_k\right|^p \le c\left\|f\right\|_{H_a^p}^p.$$

The inverse direction is more difficult since it requires the construction of a near-optimal atomic decomposition of f.

□

Modulus of smoothness

Def The ***difference operator*** Δ_h^r. For $h \in \mathbb{R}^n$ we define $\Delta_h(f,x) = f(x+h) - f(x)$. For general $r \ge 1$ we define

$$\Delta_h^r(f,x) = \underbrace{\Delta_h \circ \cdots \Delta_h}_{r}(f,x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x+kh).$$

Remarks

1. For $\Omega \subset \mathbb{R}^n$, we modify to $\Delta_h^r(f,x) := \Delta_h^r(f,x,\Omega)$, where $\Delta_h^r(f,x) = 0$, in the case $[x, x+rh] \not\subset \Omega$. So for $\Omega = [a,b]$, $\Delta_h^r(f,x) = 0$ on $[b-rh, b]$, for any function.
2. As an operator on $L_p(\Omega)$, $1 \le p \le \infty$, we have that $\|\Delta_h^r\|_{L_p \to L_p} \le 2^r$. Assume $\Omega = \mathbb{R}^n$, then

$$\|\Delta_h^r(f,\bullet)\|_p \le \sum_{k=0}^{r} \binom{r}{k} \|f(\bullet + kh)\|_p = \sum_{k=0}^{r} \binom{r}{k} \|f\|_p = 2^r \|f\|_p$$

Def The ***modulus of smoothness*** of order r of a function $f \in L_p(\Omega)$, $0 < p \le \infty$, at the parameter $t > 0$

$$\omega_r(f,t)_p := \sup_{|h| \le t} \|\Delta_h^r(f,x)\|_{L_p(\Omega)}.$$

For $r = 1$ the modulus of smoothness is called the ***modulus of continuity***.

Example non continuous function. Let $\Omega = [-1,1]$. $f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \end{cases}$

Let's compute $\omega_r(f,t)_{L_p([-1,1])}$.

$$\Delta_h(f,x) = \begin{cases} 0 & -1 \le x \le -h \\ 1 & -h < x \le 0 \\ 0 & 0 < x \le 1 \end{cases}$$

For $p = \infty$ we get $\omega_1(f,t)_{L_\infty([-1,1])} = \sup_{|h| \le t} \|\Delta_h f\|_{L_\infty([-1,1])} = 1$.

For $p \ne \infty$ we get $\omega_1(f,t)_{L_p([-1,1])} = \sup_{|h| \le t} \|\Delta_h f\|_{L_p([-1,1])} = t^{1/p}$.

$$\Delta_h^2(f,x) = \Delta_h(\Delta_h f, x) = \begin{cases} 0 & -1 \le x \le -2h \\ 1 & -2h < x \le -h \\ -1 & -h < x \le 0 \\ 0 & 0 \le x \le 1 \end{cases}$$

We get $\omega_2(f,t)_{L_p([-1,1])} = (2t)^{1/p}$

In general, we get $\omega_r(f,t)_{L_p([-1,1])} \le C(r,p) t^{1/p}$

Quick jump into the "future" (Generalized Lipschitz / Besov smoothness)… for $\alpha < 1/\tau$, $r = \lfloor \alpha \rfloor + 1$,

$$|f|_{B_{\tau,\infty}^{\alpha}} := \sup_{t>0} t^{-\alpha} \omega_r(f,t)_\tau \le \sup_{0<t\le 2} t^{-\alpha} \omega_r(f,t)_\tau \le c \sup_{0<t\le 2} t^{1/\tau - \alpha} < \infty.$$

We then say that f has α (weak-type) smoothness. Observe that in this example α can be arbitrarily large as long as the integration takes place with τ sufficiently small.

Properties

1. $\omega_r(f,t)_p \le 2^r \|f\|_{L_p(\Omega)}$, $1 \le p \le \infty$.
2. $\omega_r(f,t)_p$ is non-decreasing in t
3. For $1 \le p \le \infty$ the ***sub-linearity*** property

$$\left|\Delta_h^r(f+g,x)\right| \le \left|\Delta_h^r(f,x)\right| + \left|\Delta_h^r(g,x)\right|,$$

gives

$$\omega_r(f+g,t)_p \le \omega_r(f,t)_p + \omega_r(g,t)_p.$$

4. For $N \ge 1$, $\omega_r(f,Nt)_p \le N^r \omega_r(f,t)_p$, $1 \le p \le \infty$. We prove this using the property (**assignment**)

$$\Delta_{Nh}^r(f,x) = \sum_{k_1=0}^{N-1} \cdots \sum_{k_r=0}^{N-1} \Delta_h^r(f, x+k_1h+\cdots+k_rh).$$

Let's see the case $r=1$,

$$\begin{aligned}\Delta_{Nh}(f,x) &= f(x+Nh) - f(x)\\ &= f(x+Nh) - f(x+(N-1)h) + f(x+(N-1)h) - \cdots + f(x+h) - f(x)\\ &= \sum_{k=0}^{N-1} \Delta_h(f, x+kh)\end{aligned}$$

Then, for any $h \in \mathbb{R}^n$, $|h| \le t$

$$\begin{aligned}\left\|\Delta_{Nh}^r(f,\cdot)\right\|_p &\le \sum_{k_1=0}^{N-1} \cdots \sum_{k_r=0}^{N-1} \left\|\Delta_h^r(f, \cdot + k_1h+\cdots+k_rh)\right\|_p\\ &= \sum_{k_1=0}^{N-1} \cdots \sum_{k_r=0}^{N-1} \left\|\Delta_h^r(f,\cdot)\right\|_p \le N^r \omega_r(f,t)_p.\end{aligned}$$

Taking supremum over all $h \in \mathbb{R}^n$, $|h| \le t$, gives $\omega_r(f,Nt)_p \le N^r \omega_r(f,t)_p$. It is easy to see that for $0 < p < 1$, the same proof yields $\omega_r(f,Nt)_p \le N^{r/p} \omega_r(f,t)_p$.

5. From (4) we get for $1 \le p \le \infty$,

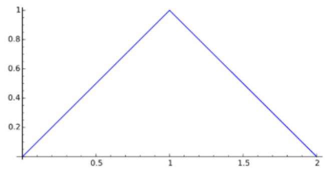
$$\omega_r(f,\lambda t)_p \le (\lambda+1)^r \omega_r(f,t)_p, \qquad \lambda > 0$$

proof $\omega_r(f,\lambda t)_p \le \omega_r(f, \lfloor \lambda+1 \rfloor t)_p \le (\lfloor \lambda+1 \rfloor)^r \omega_r(f,t)_p \le (\lambda+1)^r \omega_r(f,t)_p$.

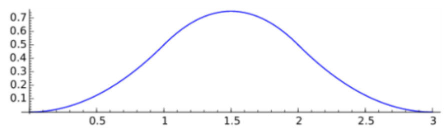
Theorem [connection between Sobolev and modulus] For $g \in W_p^r(\Omega)$, $1 \le p \le \infty$, we have that

$$\omega_r(g,t)_{L_p(\Omega)} \le C(r,n) t^r |g|_{W_p^r(\Omega)}, \qquad \forall t > 0.$$

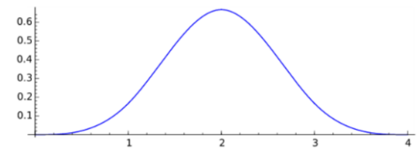
Proof for $\Omega = \mathbb{R}$. Recall the B-Splines, $N_1 = \mathbf{1}_{[0,1]^n}$. In general, $N_r := N_{r-1} * N_1 = \int_{\mathbb{R}^n} N_{r-1}(x-t) N_1(t)\,dt$.



N_2



N_3



N_4

- Properties:
  - Order r
  - Support $[0,r]^n$
  - Piecewise polynomial of degree $r-1$ with breakpoints (knots) at the integers
  - Smoothness $r-2$, thus in Sobolev W_p^{r-1}.
  - Tensor-product in multivariate case $N_r(x) := \tilde{N}_r(x_1) \times \cdots \times \tilde{N}_r(x_n)$, where $\tilde{N}_r$ is the univariate B-spline.
  - $\int_{\mathbb{R}^n} N_r(x)\,dx = 1$

Here, we use the fact that for $h \in \mathbb{R}^n$, $\left|\Delta_{-h}^r(f,x)\right| = \left|\Delta_h^r(f,x-rh)\right|$. So, w.l.g., for any $t>0$, we can work with $0 < h \le t$. Define $N_r(x,h) := h^{-1} N_r(h^{-1}x)$, $h>0$. Let $g \in C^1(\mathbb{R})$. Then

$$\begin{aligned} h^{-1}\Delta_h(g,x) &= h^{-1}\left(g(x+h) - g(x)\right) \\ &= h^{-1}\int_x^{x+h} g'(u)\,du \\ &= \int_{\mathbb{R}} g'(x+u) N_1(u,h)\,du \end{aligned}$$

We claim that for $g \in C^r(\mathbb{R})$

$$h^{-r}\Delta_h^r(g,x) = \int_{\mathbb{R}} g^{(r)}(x+u) N_r(u,h)\,du$$

To see that we apply induction

$$
\begin{aligned}
h^{-r}\Delta_h^r(g,x) &= h^{-1}h^{-(r-1)}\left(\Delta_h^{r-1}(g,x+h)-\Delta_h^{r-1}(g,x)\right)\\
&= h^{-1}\left(\int_{\mathbb{R}} g^{(r-1)}(x+h+u)N_{r-1}(u,h)\,du-\int_{\mathbb{R}} g^{(r-1)}(x+u)N_{r-1}(u,h)\,du\right)\\
&= h^{-1}\int_x^{x+h}\int_{-\infty}^{\infty} g^{(r)}(v+u)N_{r-1}(u,h)\,dudv\\
&= \int_{-\infty}^{\infty} N_{r-1}(u,h)\left(h^{-1}\int_x^{x+h} g^{(r)}(v+u)dv\right)du\\
&= \int_{-\infty}^{\infty} N_{r-1}(u,h)\left(\int_{-\infty}^{\infty} g^{(r)}(v+u)N_1(v-x,h)dv\right)du\\
&\overset{v+u=x+y}{=} \int_{-\infty}^{\infty} N_{r-1}(u,h)\left(\int_{-\infty}^{\infty} g^{(r)}(x+y)N_1(y-u,h)\,dy\right)du\\
&= \int_{-\infty}^{\infty} g^{(r)}(x+y)\left(\int_{-\infty}^{\infty} N_{r-1}(u,h)N_1(y-u,h)\,du\right)dy\\
&= \int_{-\infty}^{\infty} g^{(r)}(x+y)\mathrm{N}_r(y,h)\,\mathrm{dy}
\end{aligned}
$$

Now, let's see the proof for $p=1$. Let $0<h\le t$

$$
\begin{aligned}
\int_{\mathbb{R}}\left|\Delta_h^r(g,x)\right|dx &\le h^r\int_{\mathbb{R}}\int_{\mathbb{R}}\left|g^{(r)}(x+u)\right|\left|N_r(u,h)\right|dudx\\
&\le h^r\int_{\mathbb{R}}\left|N_r(u,h)\right|du\int_{\mathbb{R}}\left|g^{(r)}(x+u)\right|dx\\
&\le t^r\int_{\mathbb{R}}\left|g^{(r)}(x)\right|dx\\
&\le t^r\left|g\right|_{W_1^r(\mathbb{R})}.
\end{aligned}
$$

For general $1\le p<\infty$ we need Minkowski's inequality (**assignment**). It says that for measurable non-negative functions φ,ρ

$$
\left\{\int_A\left(\int_B\varphi(y)\rho(x,y)\,dy\right)^p dx\right\}^{1/p}\le\int_B\varphi(y)\left(\int_A\rho(x,y)^p\,dx\right)^{1/p}dy
$$

Or written differently

$$
\left\|\int_B\varphi(y)\rho(\cdot,y)\,dy\right\|_{L_p(A)}\le\int_B\varphi(y)\left\|\rho(\cdot,y)\right\|_{L_p(A)}dy
$$

Using it we have

$$\int_{\mathbb{R}} \left|\Delta_h^r(g,x)\right|^p dx \le h^{pr} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left|g^{(r)}(x+u)\right| \left|N_r(u,h)\right| du \right)^p dx$$
$$\le h^{pr} \left(\int_{\mathbb{R}} \left|N_r(u,h)\right| \left\|g^{(r)}(\cdot+u)\right\|_{L_p(\mathbb{R})} du \right)^p$$
$$\le h^{pr} \left(\int_{\mathbb{R}} \left|N_r(u,h)\right| \left\|g^{(r)}\right\|_{L_p(\mathbb{R})} du \right)^p$$
$$\le t^{pr} \left\|g^{(r)}\right\|_{L_p(\mathbb{R})}^p$$
$$= t^{pr} \left|g\right|_{W_p^r(\mathbb{R})}^p .$$

For a general function $g \in W_p^r(\mathbb{R})$ we use the density of $C^r(\mathbb{R}) \cap W_p^r(\mathbb{R})$ in $W_p^r(\mathbb{R})$ (**assignment**)

□

Corollary For any $P \in \Pi_{r-1}(\mathbb{R})$, $P(x) = \sum_{k=0}^{r-1} a_k x^k$,

$$h^{-r}\Delta_h^r(P,x) = \int_{\mathbb{R}} P^{(r)}(x+u) N_r(u,h) du = 0 \Rightarrow \Delta_h^r(P,x) = 0 \Rightarrow \omega_r(P,t)_p = 0$$

Marchaud inequalities

We know that for any $1 \le k < r$, $1 \le p \le \infty$,

$$\omega_r(f,t)_p = \sup_{|h|\le t} \left\|\Delta_h^r(f)\right\|_p = \sup_{|h|\le t} \left\|\Delta_h^{r-k}\Delta_h^k(f)\right\|_p \le 2^{r-k} \sup_{|h|\le t} \left\|\Delta_h^k(f)\right\|_p = 2^{r-k}\omega_k(f,t)_p .$$

The direct inverse cannot be true. If we take $\Omega = [a,b]$ and a polynomial $P \in \Pi_{r-1}$, then $\omega_r(P,t)_p = 0$, but we don't necessarily have $\omega_k(P,t)_p = 0$ for $0 \le k < r$.

Theorem. For any $1 \le k < r$, $1 \le p \le \infty$,

On $\Omega = \mathbb{R}$, $$\omega_k(f,t)_p \le ct^k \int_t^\infty \frac{\omega_r(f,s)_p}{s^{k+1}} ds , \qquad t > 0 .$$

On $\Omega = [a,b]$, $$\omega_k(f,t)_p \le ct^k \left(\int_t^{b-a} \frac{\omega_r(f,s)_p}{s^{k+1}} ds + \frac{\|f\|_p}{(b-a)^k} \right), \quad 0 < t \le \frac{b-a}{r} .$$

Proof of the case $\Omega = \mathbb{R}^n$. We prove first for $r = k+1$ and then apply induction. Using induction on k , we get that

$$Q_k(x) := \frac{1-2^{-k}(x+1)^k}{x-1} \in \Pi_{k-1} .$$

This is by

$$Q_k(x) = \frac{1-2^{-k}(x+1)^k}{x-1} = \frac{1-\frac{x+1}{2}+\frac{x+1}{2}-2^{-k}(x+1)^k}{x-1} = Q_1(x) + \frac{x+1}{2} Q_{k-1}(x)$$

This gives

$$Q_k(x)(x-1)=1-2^{-k}(x+1)^k \Rightarrow Q_k(x)(x-1)^{k+1}=(x-1)^k-2^{-k}(x^2-1)^k$$
$$\Rightarrow (x-1)^k = 2^{-k}(x^2-1)^k + Q_k(x)(x-1)^{k+1}$$

With $T_h(f,x):=f(x+h)$ we have

$$(T_h-I)^k = 2^{-k}(T_{2h}-I)^k + Q_k(T_h)(T_h-I)^{k+1}.$$

It is evident that $\|Q_k(T_h)\|_{L_p\to L_p} \le M(k)$. Therefore, with $|h|\le t$

$$\begin{aligned}
\|\Delta_h^k f\|_p &\le 2^{-k}\|\Delta_{2h}^k f\|_p + M\|\Delta_h^{k+1} f\|_p \\
&\le 2^{-k}\left(2^{-k}\|\Delta_{4h}^k f\|_p + M\|\Delta_{2h}^{k+1} f\|_p\right) + M\|\Delta_h^{k+1} f\|_p \\
&\le \cdots \\
&\le M\sum_{j=0}^{m} 2^{-jk}\|\Delta_{2^j h}^{k+1} f\|_p + 2^{-km}\|\Delta_{2^m h}^k f\|_p \\
&\le M\sum_{j=0}^{m} 2^{-jk}\omega_{k+1}(f,2^j t)_p + 2^{-k(m-1)}\|f\|_p .
\end{aligned}$$

So if we let $m\to\infty$

$$\begin{aligned}
\omega_k(f,t)_p &\le M\sum_{j=0}^{\infty} 2^{-jk}\omega_{k+1}(f,2^j t)_p \\
&= Mt^k\sum_{j=0}^{\infty}(2^j t)^{-k}\omega_{k+1}(f,2^j t)_p \\
&\le c(k)t^k\sum_{j=0}^{\infty}\int_{2^j t}^{2^{j+1}t}\frac{\omega_{k+1}(f,s)_p}{s^{k+1}}ds \\
&= c(k)t^k\int_t^{\infty}\frac{\omega_{k+1}(f,s)_p}{s^{k+1}}ds
\end{aligned}$$

Using induction

$$\begin{aligned}
\omega_k(f,t)_p &\le ct^k\int_t^{\infty}\frac{\omega_r(f,s)_p}{s^{k+1}}ds \\
&\le ct^k\int_t^{\infty}s^{r-k-1}ds\int_s^{\infty}\frac{\omega_{r+1}(f,u)_p}{u^{r+1}}du \\
&\le ct^k\int_t^{\infty}\frac{\omega_{r+1}(f,u)_p}{u^{r+1}}du\int_t^{u}s^{r-k-1}ds \\
&\le ct^k\int_t^{\infty}\frac{\omega_{r+1}(f,u)_p}{u^{r+1}}\left(u^{r-k}-t^{r-k}\right)du \\
&= ct^k\int_t^{\infty}\frac{\omega_{r+1}(f,u)_p}{u^{k+1}}du - ct^r\int_t^{\infty}\frac{\omega_{r+1}(f,u)_p}{u^{r+1}}du \\
&\le ct^k\int_t^{\infty}\frac{\omega_{r+1}(f,u)_p}{u^{k+1}}du.
\end{aligned}$$

□

The K-functional

Definition For two Banach spaces $X_1 \subset X_0$, the corresponding K-functional

$$K(f,t,X_0,X_1) := \inf_{f=f_0+f_1} \|f_0\|_{X_0} + t|f_1|_{X_1}$$

$$K\left(f,t,L_p(\Omega),W_p^r(\Omega)\right) := \inf_{g\in W_p^r(\Omega)} \|f-g\|_{L_p(\Omega)} + t|g|_{W_p^r(\Omega)}, \qquad 1\le p\le\infty .$$

Theorem [Equivalence of K-functional and modulus] For 'nice domains' $\Omega \subseteq \mathbb{R}^n$, $1\le p\le\infty$, $r\ge 1$, there exist $C_1, C_2 > 0$, such that for any $t>0$

$$C_1 K_r\left(f,t^r\right)_p \le \omega_r(f,t)_p \le C_2 K_r\left(f,t^r\right)_p .$$

It is easy to show that C_2 depends only on r, but the constant C_1 further depends on the geometry of Ω.
Proof of the easy direction Let $f\in L_p(\Omega)$ and let $g\in W_p^r(\Omega)$. Then

$$\begin{aligned}\omega_r(f,t)_p &\le \omega_r(f-g,t)_p + \omega_r(g,t)_p \\ &\le 2^r\|f-g\|_{L_p(\Omega)} + C(r)t^r|g|_{W_p^r(\Omega)} \\ &\le C(r)\left(\|f-g\|_{L_p(\Omega)} + t^r|g|_{W_p^r(\Omega)}\right)\end{aligned}$$

Taking infimum over all possible $g\in W_p^r(\Omega)$ we obtain the right-hand side.

□

Applications of K-functionals

The K-functional appears in many applications such as denoising. It provides a balance between approximation and smoothness.

1. Regularized Least Squares

$$\min_{g=\sum\alpha_k N_r(\cdot-k)} \|f-g\|_2^2 + t\left\|g^{(2)}\right\|_2^2 .$$

2. Denoising with Total Variation minimization over a bounded domain $\Omega\subset\mathbb{R}^n$

$$\min_{g\in W_2^1(\Omega)} \|f-g\|_2 + t|g|_{1,1}$$

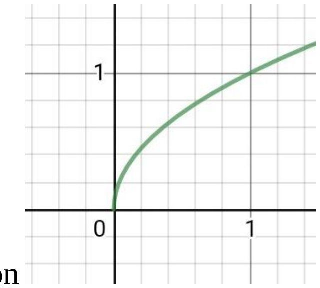
Lip spaces

Def For a domain $\Omega\subset\mathbb{R}^n$ and $0<\alpha\le 1$, we shall say that $f\in Lip(\alpha)=Lip(\alpha,\infty)$, if there exists $M>0$, such that $|f(x)-f(y)|\le M|x-y|^{\alpha}$, for all $x,y\in\Omega$. We shall denote $|f|_{Lip(\alpha)}$ by the infimum over all M satisfying the condition. Observe that we can replace the condition by

$$\left|\Delta_h(f,x)\right| \le M\left|h\right|^{\alpha},\ \forall h\in\mathbb{R}^n \Rightarrow$$
$$\omega_1(f,t)_{\infty} \le Mt^{\alpha},\ \forall t>0 \Rightarrow$$
$$t^{-\alpha}\omega_1(f,t)_{\infty} \le M,\ \forall t>0.$$

For $1\le p\le\infty$, we define

$$\left|f\right|_{lip(\alpha,p)} := \sup_{t>0} t^{-\alpha}\omega_1(f,t)_p\ .$$



Example For $f(x)=x^{\alpha},\ 0<\alpha\le 1,\ f\in Lip(\alpha),\ f\notin Lip(\beta)\ ,\ \beta>\alpha$.

Proof

Assume $f\in Lip(\beta),\ \beta>\alpha$. Then for $0<x\le 1$,

$$x^{\alpha}-0^{\alpha}=x^{\alpha}\le M(x-0)^{\beta}=Mx^{\beta}\Rightarrow x^{\alpha-\beta}\le M \Rightarrow \text{contradiction}$$

(i) We use the inequality $(a+b)^{\alpha}\le a^{\alpha}+b^{\alpha}$. Assume w.l.g $x\ge y$, we set $a=y, b=x-y$ and obtain $x^{\alpha}\le y^{\alpha}+(x-y)^{\alpha}\Rightarrow x^{\alpha}-y^{\alpha}\le(x-y)^{\alpha}$, $\left|f\right|_{Lip(\alpha)}=1$.

□

However, for any $0<\alpha\le 1,\ f(x)=x^{\alpha}\in Lip(1,1)$, because

$$\int_0^1\left|f'(x)\right|dx=1\Rightarrow f'\in L_1\Rightarrow f\in W_1^1([0,1])$$
$$\Rightarrow \omega_1(f,t)_1\le t\left|f\right|_{1,1}=t,\quad \forall t>0$$
$$\Rightarrow \left|f\right|_{Lip(1,1)}=\sup_{t>0}t^{-1}\omega_1(f,t)_1=1.$$

Generalized Lip are a special case of Besov spaces. For any $\alpha>0$, let $r:=\lfloor\alpha\rfloor+1$,

$$\left|f\right|_{B^{\alpha}_{p,\infty}} := \sup_{t>0}t^{-\alpha}\omega_r(f,t)_p\,.$$

Approximation using uniform piecewise constants (numerical integration)

The B-Spline of order one (degree zero, smoothness -1) $N_1(x)=\mathbf{1}_{[0,1]}(x)$.

Let $\Omega=\mathbb{R}$ or $\Omega=[a,b]$. We approximate from the space

$$S(N_1)^h := \left\{\sum_{k\in\mathbb{Z}}c_k N_1\left(h^{-1}x-k\right)\right\}=\left\{\sum_{k\in\mathbb{Z}}c_k\mathbf{1}_{[kh,(k+1)h]}(x)\right\}.$$

Theorem For $f\in W_p^1(\mathbb{R})$, $1\le p\le\infty$,

$$E\left(f,S(N_1)^h\right)_{L_p(\mathbb{R})} := \inf_{g\in S(N_1)^h}\left\|f-g\right\|_{L_p(\mathbb{R})}\le h\left|f\right|_{W_p^1(\mathbb{R})}.$$

Proof First assume $f\in C^1(\mathbb{R})\cap W_p^1(\mathbb{R})$. Let's take the interval $[kh,(k+1)h]$. Then, for $p=\infty$

$$\left|f(x)-f(kh)\right|=\left|\int_{kh}^{x}f'(u)\,du\right|\le h\max_{kh\le u\le(k+1)h}\left|f'(u)\right|.$$

So select $c_k := f(kh)$ and you get the theorem for $p=\infty$. For $1\le p<\infty$ we do something similar

$$\left|f(x)-f(kh)\right|^p \le \left(\int_{kh}^{(k+1)h} \left|f'(u)\right| du\right)^p, \qquad x \in \left[kh,(k+1)h\right].$$

Then

$$\int_{kh}^{(k+1)h} \left|f(x)-f(kh)\right|^p dx \le h\left(\int_{kh}^{(k+1)h} \left|f'(u)\right| du\right)^p$$

$$\underset{Holder}{\le} h\left(\|f'\|_{L_p([kh,(k+1)h])}\|1\|_{L_{p'}([kh,(k+1)h])}\right)^p \qquad 1+\frac{p}{p'} = 1+p\left(1-\frac{1}{p}\right)$$

$$= hh^{p/p'}\|f'\|^p_{L_p([kh,(k+1)h])} \qquad = 1+p-1 = p$$

$$= h^p\|f'\|^p_{L_p([kh,(k+1)h])}.$$

Therefore, with $g(x) := \sum_k f(kh)N_1(h^{-1}x-k)$, we get

$$\|f-g\|_p^p = \int_{-\infty}^{\infty}\left|f(x)-g(x)\right|^p dx = \sum_k \int_{kh}^{(k+1)h}\left|f(x)-f(kh)\right|^p dx \le \sum_k h^p\|f'\|^p_{L_p([kh,(k+1)h])} = h^p\|f'\|_p^p.$$

Now assume $f \in W_p^1(\mathbb{R})$, $1 \le p < \infty$. There exist sequences $\{f_k\}$, $f_k \in C^1(\mathbb{R}) \cap W_p^1(\mathbb{R})$, $\{g_k\}$, $g_k \in S(N_1)^h$, such that $\|f-f_k\|_{W_p^1(\mathbb{R})} \underset{k\to\infty}{\to} 0$ and $\|f_k-g_k\|_{L_p(\mathbb{R})} \le h|f_k|_{W_p^1(\mathbb{R})}$. This gives

$$\|f-g_k\|_p \le \|f-f_k\|_p + \|f_k-g_k\|_p$$

$$\le \|f-f_k\|_p + h|f_k|_{1,p} \underset{k\to\infty}{\to} 0 + h|f|_{1,p}$$

□

Linear approximation of Lip functions

Theorem: Let $f \in Lip(\alpha)$. Approximation with piecewise constants gives

$$E_N(f)_{L_\infty([0,1])} := \inf_{\phi \in S(N_1)^{1/N}}\|f-\phi\|_\infty \le CN^{-\alpha}|f|_{Lip(\alpha)}.$$

Proof [Classic technique] Recall that for $g \in C^1[0,1]$, we constructed $\phi_g \in S(N_1)^{1/N}$, such that $E_N(g)_\infty \le \|g-\phi_g\|_\infty \le N^{-1}|g|_{1,\infty}$. Therefore,

$$\|f-\phi_g\|_\infty \le \|f-g\|_\infty + \|g-\phi_g\|_\infty$$

$$\le \|f-g\|_\infty + N^{-1}|g|_{1,\infty}$$

For a sequence $\{g_k\}$, with $K_1(f,N^{-1})_\infty = \lim_{k\to\infty}\|f-g_k\|_\infty + N^{-1}|g_k|_{1,\infty}$, we get

$$\|f-\phi_{g_k}\|_\infty \le \|f-g_k\|_\infty + N^{-1}|g_k|_{1,\infty} \underset{k\to\infty}{\to} K_1(f,N^{-1})_\infty.$$

Using the equivalence of the modulus of smoothness and K-functional,

$$
\begin{aligned}
E_N(f)_\infty &\le K_1\left(f, N^{-1}\right)_\infty \\
&\le C\omega_1\left(f, N^{-1}\right)_\infty \\
&\le CN^{-\alpha} |f|_{Lip(\alpha)}.
\end{aligned}
$$

□

Inverse Theorem: Assume $E_N(f)_\infty \le MN^{-\alpha}$, $N \ge 1$. Then, $f \in Lip(\alpha)$.

Intuition $0 \le y < x \le 1$**.** Let $x = y + h$, $(N+1)^{-1} \le h \le N^{-1}$. If $x, y \in \left[kN^{-1}, (k+1)N^{-1}\right]$, then with the approximation constant approximation c_k in that interval,

$$
\begin{aligned}
|f(x) - f(y)| &\le |f(x) - c_k| + |f(y) - c_k| \\
&\le 2MN^{-\alpha} \\
&\le 2M|x-y|^\alpha
\end{aligned}
$$

However, since they might not fall in the same interval, there is a mixing argument.

So linear approximation is kind of limited when α is small. The problem is that we're not spending enough 'budget' in the vicinity of zero.

Besov Spaces

Continuous definition

Let $\alpha > 0$, $0 < q, p \le \infty$. Let $r \ge \lfloor \alpha \rfloor + 1$. The Besov space $B_q^\alpha\left(L_p(\Omega)\right)$ is the collection of functions $f \in L_p(\Omega)$ for which

$$
|f|_{B_q^\alpha(L_p(\Omega))} := \begin{cases} \left(\int_0^\infty \left[t^{-\alpha}\omega_r(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{-\alpha}\omega_r(f,t)_p, & q = \infty. \end{cases}
$$

is finite. The norm is

$$
\|f\|_{B_q^\alpha(L_p(\Omega))} := \|f\|_{L_p(\Omega)} + |f|_{B_q^\alpha(L_p(\Omega))}.
$$

Theorem The space $B_q^\alpha\left(L_p(\Omega)\right)$ does not depend on the choice of $r \ge \lfloor \alpha \rfloor + 1$ (application of the Marchaud inequality).

Proof For $\Omega = \mathbb{R}^n$, $1 \le q < \infty$. Let $r_2 > r_1 \ge \lfloor \alpha \rfloor + 1$. We already know that for $1 \le p \le \infty$, and any $t > 0$, $\omega_{r_2}(f,t)_p \le 2^{r_2 - r_1}\omega_{r_1}(f,t)_p$ (for $0 < p \le 1$ with a different constant), so

$$
\int_0^\infty \left[t^{-\alpha}\omega_{r_2}(f,t)_p\right]^q \frac{dt}{t} \le c\int_0^\infty \left[t^{-\alpha}\omega_{r_1}(f,t)_p\right]^q \frac{dt}{t}.
$$

The other direction requires the Marchaud inequality

$$\int_0^\infty \left[t^{-\alpha}\omega_{r_1}\left(f,t\right)_p\right]^q \frac{dt}{t} \le c\int_0^\infty \left[t^{r_1-\alpha}\int_t^\infty \frac{\omega_{r_2}\left(f,s\right)_p}{s^{r_1+1}}ds\right]^q \frac{dt}{t}.$$

Denote $\theta := r_1-\alpha > 0$, and $\phi(s) := s^{-r_1}\omega_{r_2}\left(f,s\right)_p$. Then, we can apply the Hardy inequality [DL Theorem 2.3.1] for $1 \le q < \infty$, to the right-hand side

$$\begin{aligned}\int_0^\infty \left[t^{r_1-\alpha}\int_t^\infty \frac{\omega_{r_2}\left(f,s\right)_p}{s^{r_1+1}}ds\right]^q \frac{dt}{t} &= \int_0^\infty \left[t^{\theta}\int_t^\infty \frac{\phi(s)}{s}ds\right]^q \frac{dt}{t} \\ &\underset{\text{Hardy}}{\le} \frac{1}{\theta^q}\int_0^\infty \left[t^\theta\phi(t)\right]^q \frac{dt}{t} \\ &= \frac{1}{\left(r_1-\alpha\right)^q}\int_0^\infty \left[t^{r_1-\alpha}t^{-r_1}\omega_{r_2}\left(f,t\right)_p\right]^q \frac{dt}{t} \\ &= c\int_0^\infty \left[t^{-\alpha}\omega_{r_2}\left(f,t\right)_p\right]^q \frac{dt}{t}\end{aligned}$$

□

Why are we asking for the condition $r \ge \lfloor \alpha \rfloor + 1$? Otherwise the space is ‘trivial’

Theorem (univariate case) For $r < \alpha$, $1 \le p \le \infty$, we get that $B_q^\alpha\left(L_p\left(\Omega\right)\right) = \Pi_{r-1}$ if $\Omega = [a,b]$ and $B_q^\alpha\left(L_p\left(\Omega\right)\right) = \{0\}$ if $\Omega = \mathbb{R}$.

Proof (sketch, see Proposition 2.7.1 in CA) If $f \in B_q^\alpha\left(L_p\left(\Omega\right)\right)$, then $t^{-\alpha}\omega_r\left(f,t\right)_p \le C$, for $0 < t \le t_0 < 1$. This implies that

$$t^{-r}\omega_r\left(f,t\right)_p = t^{\alpha-r}t^{-\alpha}\omega_r\left(f,t\right)_p \le Ct^{\alpha-r} \underset{t\to 0}{\to} 0.$$

The condition $t^{-r}\omega_r\left(f,t\right)_p \underset{t\to 0}{\to} 0$, in turn gives that if $f \in C^r$, then $f^{(r)} = 0$ and so $f \in \Pi_{r-1}$. If $f \notin C^r$, we use density again.

□

Theorem For a bounded domain we can equivalently integrate the semi-norm on $[0,1]$. That is,

$$|f|_{B_q^\alpha\left(L_p(\Omega)\right)} \sim \begin{cases}\left(\int_0^1 \left[t^{-\alpha}\omega_r\left(f,t\right)_p\right]^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{0<t\le 1} t^{-\alpha}\omega_r\left(f,t\right)_p, & q = \infty.\end{cases}$$

Proof If Ω is bounded, then we have $\omega_r\left(f,t\right)_p \equiv const$ for $t \ge diam(\Omega)$. Therefore for $1/2 \le t \le \infty$,

$$\omega_r\left(f,1/2\right)_p \le \omega_r\left(f,t\right)_p \le \omega_r\left(f,diam(\Omega)\right)_p = \omega_r\left(f,\frac{2diam(\Omega)}{2}\right)_p \le \left(1+2diam(\Omega)\right)^r \omega_r\left(f,1/2\right)_p.$$

This gives

$$\begin{aligned}\int_1^\infty \left[t^{-\alpha}\omega_r\left(f,t\right)_p\right]^q \frac{dt}{t} &\le C\left(\omega_r\left(f,1/2\right)_p\right)^q \int_1^\infty t^{-q\alpha-1}dt \\ &\le C\left(\omega_r\left(f,1/2\right)_p\right)^q \\ &\le C\left(\alpha,q,\Omega\right)\int_{1/2}^1 \left[t^{-\alpha}\omega_r\left(f,t\right)_p\right]^q \frac{dt}{t}\end{aligned}$$

□

Lemma For any domain taking the integral over $[0,1]$ gives a quasi-norm equivalent to $\|f\|_{B_q^\alpha(L_p(\Omega))}$

Proof We replace the integral over $[1,\infty]$ by

$$\int_1^\infty \left[t^{-\alpha}\omega_r(f,t)_p\right]^q \frac{dt}{t} \le C\|f\|_p^q \int_1^\infty t^{-q\alpha-1}dt$$
$$= C(\alpha,q)\|f\|_p^q .$$

Therefore

$$\|f\|_{B_q^\alpha(L_p(\Omega))} \sim \|f\|_p + \left(\int_0^1 \left[t^{-\alpha}\omega_r(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q} .$$

□

Theorem $B_{q_1}^{\alpha_1}(L_p) \subseteq B_{q_2}^{\alpha_2}(L_p)$ if $\alpha_2 < \alpha_1$.

Proof $(q_1 = q_2)$ We may use $r_1 = \lfloor \alpha_1 \rfloor + 1 \ge \lfloor \alpha_2 \rfloor + 1 = r_2$ to equivalently define $B_{q_2}^{\alpha_2}(L_p)$

For $0 < t \le 1$, $t^{-\alpha_2} \le t^{-\alpha_1}$. So,

$$\|f\|_{B_q^{\alpha_2}(L_p)} \le C\left(\|f\|_p + \left(\int_0^1 \left[t^{-\alpha_2}\omega_{r_1}(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q}\right)$$
$$\le C\left(\|f\|_p + \left(\int_0^1 \left[t^{-\alpha_1}\omega_{r_1}(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q}\right)$$
$$\le C\|f\|_{B_q^{\alpha_1}(L_p)}$$

□

Theorem $W_p^m \subseteq B_q^\alpha(L_p)$, $\forall \alpha < m$, $1 \le p \le \infty$, $0 < q \le \infty$.

Proof Let $g \in W_p^m(\Omega)$. This implies $g \in L_p(\Omega)$. We have that $r := \lfloor \alpha \rfloor + 1 \le m$. It is sufficient to take the integral over $[0,1]$.

$$\int_0^1 \left[t^{-\alpha}\omega_r(g,t)_p\right]^q \frac{dt}{t} \le C\int_0^1 \left[t^{-\alpha}t^r |g|_{r,p}\right]^q \frac{dt}{t}$$
$$\le C|g|_{r,p}^q \int_0^1 t^{(r-\alpha)q-1}dt$$
$$\le C|g|_{r,p}^q .$$

□

Discretization of the Besov semi-norm

Theorem One has the following equivalent form of the Besov semi-norm

$$|f|_{B_q^\alpha(L_p(\Omega))} \sim \begin{cases} \left(\sum_{k=-\infty}^{\infty} \left[2^{k\alpha}\omega_r(f,2^{-k})_p\right]^q\right)^{1/q}, & 0 < q < \infty. \\ \sup_{k\in\mathbb{Z}} 2^{k\alpha}\omega_r(f,2^{-k})_p, & q = \infty. \end{cases}$$

Proof Define $\varphi(t) := t^{-\alpha}\omega_r(f,t)_p$. Then we claim that for $t \in [2^{-k-1}, 2^{-k}]$, $k \in \mathbb{Z}$, we have

$$2^{-r}\varphi(2^{-k}) \le \varphi(t) \le 2^{\alpha}\varphi(2^{-k}).$$

To see that, we use the following properties:

(i) $\omega_r(f,t)_p$ is non-decreasing

(ii) For $N \in \mathbb{N}$, $1 \le p \le \infty$, $\omega_r(f,Nt)_p \le N^r \omega_r(f,t)_p$

The left-hand side

$$2^{-r}\varphi(2^{-k}) = 2^{k\alpha - r}\omega_r(f,2^{-k})_p = 2^{k\alpha-r}\omega_r(f,22^{-k-1})_p$$
$$\underset{\text{(ii)}}{\le} 2^{k\alpha-r}2^r\omega_r(f,2^{-k-1})_p \underset{\text{(i)}}{\le} 2^{k\alpha}\omega_r(f,t)_p \le t^{-\alpha}\omega_r(f,t)_p$$

The right-hand side

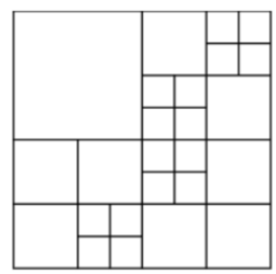
$$t^{-\alpha}\omega_r(f,t)_p \underset{\text{(i)}}{\le} t^{-\alpha}\omega_r(f,2^{-k})_p \le 2^{(k+1)\alpha}\omega_r(f,2^{-k})_p \le 2^{\alpha}\varphi(2^{-k})$$

This gives us for $0 < q < \infty$, $k \in \mathbb{Z}$

$$\int_{2^{-k-1}}^{2^{-k}} \varphi(t)^q \frac{dt}{t} \sim \varphi(2^{-k})^q \int_{2^{-k-1}}^{2^{-k}} \frac{dt}{t} \sim \varphi(2^{-k})^q \Rightarrow \int_{2^{-k-1}}^{2^{-k}} \left(t^{-\alpha}\omega_r(f,t)_p\right)^q \frac{dt}{t} \sim \left[2^{k\alpha}\omega_r(f,2^{-k})_p\right]^q.$$

□

Discretization over cubes



Definition [Dyadic cubes] Let $D := \{D_k : k \in \mathbb{Z}\}$

$$D_k := \left\{Q = 2^{-kn}[m_1, m_1+1] \times \cdots \times [m_n, m_n+1] : m \in \mathbb{Z}^n\right\}.$$

Observe that $Q \in D_k \Rightarrow |Q| = 2^{-kn}$.

For nonlinear/adaptive/sparse approximation in $L_p(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, it is useful to use the special cases of Besov spaces

$$B_\tau^\alpha := B_\tau^\alpha(L_\tau(\Omega)), \qquad \frac{1}{\tau} = \frac{\alpha}{n} + \frac{1}{p}.$$

Theorem $\Omega = \mathbb{R}^n$. We have the equivalence

$$|f|_{B_\tau^\alpha} \sim \left(\sum_{k \in \mathbb{Z}} \left(2^{k\alpha}\omega_r(f,2^{-k})_\tau\right)^\tau\right)^{1/\tau} \sim \left(\sum_{Q \in D} \left(|Q|^{-\alpha/n}\omega_r(f,Q)_\tau\right)^\tau\right)^{1/\tau},$$

$$\omega_r(f,Q)_\tau := \sup_{h \in \mathbb{R}^n} \left\|\Delta_h^r(f,Q,\cdot)\right\|_{L_\tau(Q)}.$$

For $\Omega = [0,1]^n$, with $l(Q)$ denoting the level of the cube Q, we may take the sum over $k \ge 0$

$$|f|_{B_\tau^\alpha} \sim \left(\sum_{k=0}^{\infty} \left(2^{k\alpha}\omega_r(f,2^{-k})_\tau\right)^\tau\right)^{1/\tau} \sim \left(\sum_{Q \in D, l(Q) \ge 0} \left(|Q|^{-\alpha/n}\omega_r(f,Q)_\tau\right)^\tau\right)^{1/\tau}.$$

The following theorem generalizes what we showed for the univariate case

Theorem Let $f(x)=\mathbf{1}_{\tilde{\Omega}}(x)$, $\tilde{\Omega}\subset[0,1]^n$, a domain with smooth boundary. Then $f\in B_\tau^\alpha$, $\alpha<1/\tau$.

Proof For any Q, we have that $\omega_r(f,Q)_\tau=0$, if $\partial\tilde{\Omega}\cap Q=\varnothing$. Otherwise, if $l(Q)=k$,

$$\omega_r(f,Q)_\tau\le\left(\int_Q 1^\tau\right)^{1/\tau}=|Q|^{1/\tau}=2^{-kn/\tau}.$$

Therefore,

$$\begin{aligned}|f|_{B_\tau^\alpha}^\tau&\le C\sum_{l(Q)\ge 0}\left(|Q|^{-\alpha/n}\omega_r(f,Q)_\tau\right)^\tau\\&\le C\sum_{k=0}^\infty\left(2^{k\alpha}2^{-kn/\tau}\right)^\tau\#\left\{Q:l(Q)=k,\ Q\cap\partial\tilde{\Omega}\ne\varnothing\right\}\\&=C\sum_{k=0}^\infty 2^{k(\alpha\tau-n)}\#\left\{Q:l(Q)=k,\ Q\cap\partial\tilde{\Omega}\ne\varnothing\right\}\end{aligned}$$

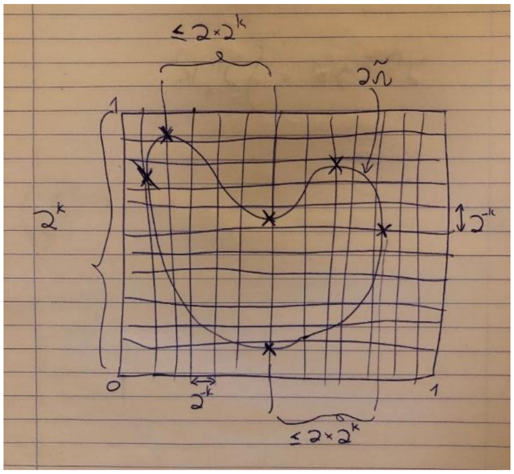
We argue that

$$\#\left\{Q:l(Q)=k,\ Q\cap\partial\tilde{\Omega}\ne\varnothing\right\}\le c(\tilde{\Omega})2^{k(n-1)}.\quad(*)$$

This implies that if $\alpha<1/\tau$

$$|f|_{B_\tau^\alpha}^\tau\le C\sum_{k=0}^\infty 2^{k(\alpha\tau-n)}2^{k(n-1)}=C\sum_{k=0}^\infty 2^{k(\alpha\tau-1)}<\infty.$$

Let's get back to the estimate (*). Let use show a picture argument for $\tilde{\Omega}\subset[0,1]^2$. There is a finite number of points where the gradient of the boundary of the domain is aligned with one of the main axes. Between these points, the boundary segments are monotone in x_1 and x_2, and therefore can only intersect at most 2×2^k dyadic cubes.



Interpolation spaces

For $0<\theta<1$, $0<q\le\infty$, $X,Y(r)$,

$$K(f,t):=K(X,Y,f,t):=\inf_{g\in Y}\left\{\|f-g\|_X+t|g|_Y\right\}.$$

$$|f|_{(X,Y)_{\theta,q}}=|f|_{\theta,q}:=\begin{cases}\left(\int_0^1\left[t^{-\theta}K(f,t)\right]^q\frac{dt}{t}\right)^{1/q}, & 0<q<\infty,\\ \sup_{0<t\le 1}t^{-\theta}K(f,t), & q=\infty.\end{cases}$$

$$\|f\|_{\theta,q}:=\|f\|_X+|f|_{\theta,q}.$$

It is convenient to discretize at $t_m = \left(2^r\right)^{-m} = 2^{-mr}$, $m \ge 0$,

$$|f|_{\theta,q} \sim \begin{cases} \left(\sum_{m=0}^{\infty} \left[2^{mr\theta} K\left(f, 2^{-mr}\right) \right]^q \right)^{1/q}, & 0 < q < \infty, \\ \sup_{m \ge 0} 2^{mr\theta} K\left(f, 2^{-mr}\right), & q = \infty. \end{cases}$$

Def We call the space of functions for which $\|f\|_{\theta,q}$ is finite the **interpolation space** $(X,Y)_{\theta,q}$.

Observe that by definition $(X,Y)_{\theta,q} \subset X$, while for $0 < \theta < 1$, we have that $Y \subset (X,Y)_{\theta,q}$. To see this, let $g \in Y$. Then,

$$\begin{aligned} \sum_{m=0}^{\infty} \left[2^{mr\theta} K\left(g, 2^{-mr}\right) \right]^q &\le \sum_{m=0}^{\infty} \left[2^{mr\theta} 2^{-mr} |g|_Y \right]^q \\ &= |g|_Y^q \sum_{m=0}^{\infty} 2^{mrq(\theta-1)} \\ &\le C(r,q,\theta) |g|_Y^q. \end{aligned}$$

Also, for $\theta_2 \le \theta_1$, $(X,Y)_{\theta_1,q} \subseteq (X,Y)_{\theta_2,q}$. So, for $0 < \theta < 1$, we have a 'scale' of spaces between Y and X .

Theorem $\left(L_p, W_p^r\right)_{\theta,q} = B_q^{\alpha}\left(L_p\right)$, $\alpha = \theta r$, $0 < \theta < 1$, $1 \le p \le \infty$.

Proof Observe first that $f \in \left(L_p, W_p^r\right)_{\theta,q} \Rightarrow f \in L_p$. Now, for $0 < q < \infty$, it is sufficient to bound the Besov semi-norm integral over $[0,1]$

$$\begin{aligned} \int_0^1 \left[t^{-\theta} K(f,t) \right]^q \frac{dt}{t} &= \int_0^1 \left[t^{-\theta} K_r\left(f,t\right)_p \right]^q \frac{dt}{t} \\ &\sim \int_0^1 \left[t^{-\alpha/r} \omega_r\left(f, t^{1/r}\right)_p \right]^q \frac{dt}{t} \qquad s = t^{1/r} \Rightarrow ds = \frac{1}{r} t^{1/r-1} dt \Rightarrow s^{-1} ds = \frac{1}{r} t^{-1} dt \\ &\underset{s=t^{1/r}}{\sim} \int_0^1 \left[s^{-\alpha} \omega_r\left(f,s\right)_p \right]^q \frac{ds}{s}. \end{aligned}$$

□

Theorem [Reiteration theorem] If

$$B_\tau^\alpha := B_\tau^\alpha\left(L_\tau(\Omega)\right), \qquad \frac{1}{\tau} = \frac{\alpha}{n} + \frac{1}{p}, \quad \frac{1}{q} = \frac{\theta\alpha}{n} + \frac{1}{p}, \quad 0 < \theta < 1,$$

then

$$\left(L_p, B_\tau^\alpha\right)_{\theta,q} \sim B_q^{\alpha\theta}.$$